2011

Question

Section: A

Question: 1 [1]

If f : R $\rightarrow$ R is defined by f(x) = 3x + 2, define f[f(x)].

Answer:

f(x) = 3x+2

f(f(x)) = 3(3x+2) + 2 = 9x+6+2 = 9x+8

Question: 2

Write the principal value of tan-1($^{-1}$). [1]

Answer:

Let, $\tan^{-1}(-1) = y$

Then, tan y = -1 = - tan $\left(\frac{\pi}{4}\right)$ = tan $\left(-\frac{\pi}{4}\right)$

We know that the range of the principal value branch of $\tan^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and tan $\left(-\frac{\pi}{4}\right) = -1$

Therefore, the principal value of $\tan^{-1}(-1)$ is $-\left(\frac{\pi}{4}\right)$

Question: 3 [1]

Write the values of x – y + z from the following equation, $\begin{bmatrix} x+y+z \\ x+z \\ y+z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 7 \end{bmatrix}$

Answer:

x + y + z = 9, x + z =5, y + z = 7

Let us solve these equations to solve for x, y, and z.

y+5 = 9

y= 4

4+z = 7

z = 3

x+4+3 = 9

x= 2

Question: 4 [1]

Write the order of the product matrix: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$

Answer:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1\times 2 & 1\times 3 & 1\times 4 \\ 2\times 2 & 2\times 3 & 2\times 4 \\ 3\times 2 & 3\times 3 & 3\times 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 & 4 \\ 4 & 6 & 8 \\ 6 & 9 & 12 \end{bmatrix}$$

The above given matrix has 3 rows, and 3 columns. Order of matrix $\Rightarrow 3 \times 3$

Question: 5 [1]

If $\begin{bmatrix} x & x \\ 1 & x \end{bmatrix} = \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix}$, write the positive value of x.

Answer:

We have $\begin{bmatrix} x & x \\ 1 & x \end{bmatrix} = \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix}$

$\Rightarrow x^2$ -x = 6 -4 $\Rightarrow x^2$ -x -2 = 0 $\Rightarrow x^2$ - 2x + x -2 = 0 $\Rightarrow$ x(x -2) +1(x - 2) = 0 $\Rightarrow$ (x - 2) (x +1) = 0

$\Rightarrow$ x = 2 or x = -1 (Not accepted) $\Rightarrow$ x = 2

Question: 6 [1]

Evaluate: $\int \frac{(1+\log x)^2}{x}$

Answer:

Let $I = \int \left\{ \frac{(1+\log x)^2}{x} \right\} \times dx$

$\left(\frac{1}{x}\right)dx = dz \Rightarrow I = \int z^2 dz = \frac{z^3}{3} + C = \left\{\frac{1}{3} \times (1+\log x)^3\right\} + C$

Question: 7 [1]

Evaluate: $\int_{`}^{\sqrt{3}} \frac{dx}{1+x^2}$

Answer:

$\frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$

$I = \int^{\sqrt{3}} \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_1^{\sqrt{3}} \left[\left\{Q \times \frac{d}{dx} \times \left(\tan^{-1} x\right)\right\} = \frac{1}{1+x^2}\right] = \tan^{-1}\left(\sqrt{3}\right) - \tan^{-1}(1)$

Question: 8 [1]

Write the position vector of the mid-point of the vector joining the points P(2,3,4), and Q(4,1,-2).

Answer:

Let, $\overrightarrow{OP} = 2\hat{i} + 3\hat{j} + 4\hat{k}$, and $\overrightarrow{OQ} = 4\hat{i} + \hat{j} - 2\hat{k}$

Mid – point of two points is given by, $\frac{\overrightarrow{OP} + \overrightarrow{OQ}}{2}$

$\Rightarrow \frac{2\hat{i} + 3\hat{j} + 4k + 4\hat{i} + \hat{j} - 2k}{2} \Rightarrow \frac{6\hat{i} + 4\hat{j} - 2k}{2}$

$\Rightarrow 3\hat{i} + 2\hat{j} - 2k$

$\Rightarrow \overrightarrow{OR} = 3\hat{i} + 2\hat{j} - k$

Question: 9 [1]

If $\left(\vec{a}.\vec{a}\right) = 0$, and $\left(\vec{a}.\vec{b}\right) = 0$, then what can be concluded about the vector $\vec{b}$?

Answer:

$\vec{a}.\vec{a} = 0$ if $\vec{a}$ is perpendicular to $\vec{b}$, $\vec{a}.\vec{a} = 0$ if $\vec{a} = 0$, $\vec{a}.\vec{a} = 0$, and $\vec{a}.\vec{b} = 0$. Then this implies if $\vec{a} = 0$, then $\vec{b}$ is any vector. Hence we can conclude that $\vec{b}$ is any vector.

Question: 10 [1]

What are the direction cosines of a line, which makes equal angles with the co-ordinate axes?

Answer:

We know sum of the squares of the direction cosines is one, i.e., $\cos^2\alpha + \cos2\beta + \cos^2\gamma = 1$
But it is given that, $\alpha = \beta = \gamma\alpha = \beta = \gamma$. Therefore,
$\cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1 \Rightarrow 3\cos^2\alpha = 1$, and

$\cos^2\alpha = \frac{1}{3}$ =>3$\cos\alpha = \pm \frac{1}{\sqrt{3}}$

Hence the direction cosines are, $\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$

Section: B

Question: 11

[4]

Consider f: R+ → [4, ∞] given by f(x) = x^2 + 4. Show that f is invertible with the inverse (f-1) of f given by f-1(y) = $\sqrt{y-4}$, where R is the set of all non-negative real numbers.

Answer:

$f : R_+ \rightarrow [4, \infty)$ is given as $f(x) = x^2 + 4$.

One-one

Let $f(x) = f(y) \Rightarrow x^2 + 4 = y^2 + 4 \Rightarrow x^2 = y^2 \Rightarrow x = y$ [as $x = y = y \in R_+$]

∴ f is a one-one function.

For $y \in [4, \infty)$, let $y = x^2 + 4$.

$\Rightarrow x^2 = y - 4 \geq 0$ [as $y \geq 4$]

$\Rightarrow x = \sqrt{y-4} \geq 0$

Therefore, for any $y \in R$, there exists $x = \sqrt{y-4} \in R$ such that

$f(x) = f\left(\sqrt{y-4}\right) = \left(\sqrt{y-4}\right)^2 + 4 = y - 4 + 4 = y$

∴ f is onto.

Thus, f is one-one, onto, where, f - 1 exists. Let us define, g: [4, ∞) → R+ byg (y) = $\sqrt{y-4}$

Now, $gof(x) = g(f(x)) = g(x^2+4) = \sqrt{x^2+4} = \sqrt{x^2} = x$, and, $fog(y) = f\{g(y)\} = f\left(\sqrt{y-4}\right)$

$= \left(\sqrt{y-4}\right)^2 + 4 = (y-4) = y$

$\therefore gof = fo\, g = 1_R$.

Hence, f is invertible, and the inverse of f is given by $f^{-1}(y) = g(y) = \sqrt{y-4}$

Question: 12

[4]

Prove $\frac{9\pi}{8} - \left\{\frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)\right\} = \frac{9}{4} \times \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$

Answer:

Given, $\left[\frac{9\pi}{8} - \left\{\frac{9}{4} \times \sin^{-1}\left(\frac{1}{3}\right)\right\}\right] = \frac{9}{4} \times \left\{\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)\right\}$

Rearranging the terms, we need to prove that, $\frac{\pi}{8} = \frac{1}{4} \times \left\{\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) + \sin^{-1}\left(\frac{1}{3}\right)\right\}$

$$\text{R.H.S.} = \frac{1}{4} \times \left[\left\{\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)\right\} + \left\{\sin^{-1}\left(\frac{1}{3}\right)\right\}\right]$$

We know that, $\sin^{-1x} + \sin^{-1} y = \sin^{-1}\left\{\left(x\sqrt{1-y^2}\right)+\left(y\sqrt{1-x^2}\right)\right\}$

By taking, $x = \frac{2\sqrt{2}}{3}$, and $y = \frac{1}{3}$

$\sqrt{1-y^2} = \sqrt{1-\frac{1}{9}} = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$, and $\sqrt{1-x^2} = \sqrt{1-\frac{8}{9}} = \frac{1}{3}$

$\Rightarrow$ R.H.S. $= \frac{1}{4}\times\sin^{-1}\left\{\left(\frac{2\sqrt{2}}{3}\sqrt{1-\frac{1}{9}}\right)+\left(\frac{1}{3}\sqrt{1-\frac{8}{9}}\right)\right\} = \frac{1}{4}\sin^{-1}\left\{\left(\frac{2\sqrt{2}}{3}\times\frac{2\sqrt{2}}{3}\right)+\left(\frac{1}{3}\times\frac{1}{3}\right)\right\}$

$= \frac{1}{4}\times\left\{\sin^{-1}\left(\frac{8}{9}+\frac{1}{9}\right)\right\}$

or, $\frac{1}{4}\left(\sin^{-1}1\right) = \left(\frac{1}{4}\times\frac{\pi}{2}\right) = \frac{\pi}{8} = \text{L.H.S}$

OR

Solve the following equation for x: $\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1}(x), x > 0$

Answer:

Let $x = \tan\theta$, $\Rightarrow \theta = \tan^{-1} x$ $x = \tan\theta$, $\Rightarrow \theta = \tan^{-1} x$

Substituting the value of x in the given equation, we have $\tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right) = \frac{1}{2}\times\left\{\left(\tan^{-1}\right)\times(\tan\theta)\right\}$

We know that if,

$\tan\left(\frac{\pi}{4}\right) = 1$, then

$\Rightarrow \tan^{-1}\left(\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}-\tan\theta}\right) = \frac{1}{2}\theta \Rightarrow \tan^{-1}\tan\left(\frac{\pi}{4}-\theta\right) = \frac{1}{2}\theta \Rightarrow \frac{\pi}{4}-\theta = \frac{\theta}{2} \Rightarrow \frac{30}{2} = \frac{\pi}{4} \Rightarrow \theta = \frac{\pi}{6} = \tan^{-1}x$

$\Rightarrow x = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

Question: 13 [4]

Prove, using properties of $\begin{vmatrix} (y+k) & y & y \\ y & (y+k) & y \\ y & y & (y+k) \end{vmatrix} = k^2(3y+k)$

Answer:

<u>Step:</u> 1

Let $\Delta = \begin{vmatrix} (y+k) & y & y \\ y & (y+k) & y \\ y & y & (y+k) \end{vmatrix}$

Take (3y+k) as the common factor from R_1

$$\Delta=(3y+k)\begin{vmatrix}1 & 1 & 1\\ y & (y+k) & y\\ y & y & (y+k)\end{vmatrix}$$

***Step:* 2**

By applying $C_1 \to C_1 - C_2 C_1 \to C_1 - C_2$, and $C_2 \to C_2 - C_3$ we get, $\Delta=(3y+k)\begin{vmatrix}0 & 0 & 1\\ -k & k & y\\ 0 & -k & (y+k)\end{vmatrix}$

By applying $C_1 \to C_1 + C_2$ we get, $\Delta=(3y+k)\begin{vmatrix}0 & 0 & 1\\ 0 & k & y\\ -k & -k & (y+k)\end{vmatrix}$

Now expanding along R_1R_1 we get,Δ= (3y+k)1(0 x−k−k×−k) =$(3y+k)k^2$

Hence, $\begin{vmatrix}(y+k) & y & y\\ y & (y+k) & y\\ y & y & (y+k)\end{vmatrix}=k^2(3y+)k$

Question: 14 [4]

Find the value of k so that the function f is defined by $f(x)=\begin{cases}\left(\dfrac{k\cos x}{\pi-2x}\right) & \text{if } x\neq\dfrac{\pi}{2}\\ 3 & \text{if } x\neq\dfrac{\pi}{2}\end{cases}$ is continuous at

$x=\dfrac{\pi}{2}$

Answer:

Since, $f(x)$is continuous at, $x=\dfrac{\pi}{2}\times f\left(\dfrac{\pi}{2}\right)=\lim\limits_{x\to\frac{\pi}{2}} f(x)=\lim\limits_{x\to\frac{\pi}{2}}=\dfrac{K\cos x}{\pi-2x}$

Here we need not find left hand, and right hand separately because f(x)f(x) is not different when,

$x<\dfrac{\pi}{2}$, and $x>\dfrac{\pi}{2}$ => $\lim\limits_{h\to 0}\left\{\dfrac{k\cos\left(\dfrac{\pi}{2}+h\right)}{\pi-2\left(\dfrac{\pi}{2}+h\right)}\right\}$

Putting, $x=\dfrac{\pi}{2}+h,\ x\to\dfrac{\pi}{2}h\to 0=\lim\limits_{h\to 0}\dfrac{-K\sin h}{-2h}=\dfrac{K}{2}\times\lim\limits_{h\to 0}\left(\dfrac{\sin h}{h}\right)=\left(\dfrac{K}{2}\times 1\right)=\dfrac{K}{2}$

$\therefore\dfrac{K}{2}=3$

or, k = 6.

Question: 15 [4]

Find the intervals in which the function f given by f(x) = sin x + cos x, $0\le x\le 2\pi$ is strictly increasing or strictly decreasing.

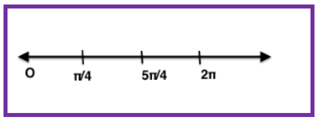
Answer:
Step: 1

f(x) = sin x + cos x
f' (x) = cos x - sin x
When f' (x) = 0, cos x - sin x=0, cos x = sin x

Given that, $x = \frac{\pi}{4}, \frac{5\pi}{4}$ as $0 \le x \le 2x$

The points, $x = \frac{\pi}{4}$, and $x = \frac{5\pi}{4}$ divides the interval $[0, 2\pi]$ into three disjoint intervals, i.e., $\left\{\left(0, \frac{\pi}{4}\right), \left(\frac{\pi}{4}, \frac{5\pi}{4}\right), \text{and} \left(\frac{5\pi}{4}, 2\pi\right)\right\}$

Step: 2



$f'(x) > 0$ if $x \in \left\{\left(0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right)\right\}$ or, f is strictly increasing in the intervals $\left\{\left(0, \frac{\pi}{4}\right), \text{and} \left(\frac{5\pi}{4}, 2\pi\right)\right\}$

Also, $f'(x) < 0$ if $x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$, and f is strictly decreasing in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

Step: 3

interval	Sign of f'(x)	Nature of function
$0, \frac{\pi}{4}$	> 0	F is strictly increases
$\frac{\pi}{4}, \frac{5\pi}{4}$	< 0	F is strictly decreases
$\frac{5\pi}{4}, 2\pi$	>0	F is strictly increases

OR

Find the points on the curve $y = x^3$ at which the slope of the tangent is equal to the y-coordinate of the point.

Answer:
Let p $(x_1.y_1)$ be a required point.

The given curve is $y = x^3$

$\therefore \frac{dy}{dx} = 3x^2 \Rightarrow \left(\frac{dy}{dx}\right)p = 3x_1^2$

Slope of tangent to the given curve at $p = 3x_1^2$

Since the slope of tangent is equal to y-coordinate of the point, $3x_1^2 = y_1$ (ii)

As the point p (x_1, y_1) lies on the curve (i), $y_1 = x_1^3$ ……………………………..…………….(iii)

From (ii), and (iii) we get, $x_1^3 = x_1^2 \Rightarrow x_1^2(x_1 - 3) = 0 \Rightarrow x_1 = 0.3$

From (iii), when $x_1=0$, $y_1 = 0$; when $x_1= 3$, $y_1 = 27$. The required points are (0,0), (3,27),

Question: 16

Prove that:

$$\frac{d}{dx}\left\{\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right\}=\sqrt{a^2-x^2}$$

Answer:

$$\text{L.H.S.}=\frac{d}{dx}\left(\frac{x}{2}\sqrt{a^2-x^2}\right)+\frac{d}{dx}\left\{\left(\frac{a^2}{2}\right)\times\sin^{-1}\left(\frac{x}{a}\right)\right\}$$

$$=\frac{1}{2}\left\{x\left(\frac{1}{2\sqrt{a^2-x^2}}\right)x-2x+\sqrt{a^2-x^2}\right\}+\left\{\frac{a^2}{2}\times\frac{1}{\sqrt{\left(1-\frac{x^2}{a^2}\right)}}\times\left(\frac{1}{a}\right)\right\}$$

$$=\left\{\left(\frac{-x^2}{2\sqrt{a^2-x^2}}\right)+\left(\frac{\sqrt{a^2-x^2}}{2}\right)+\left(\frac{a^2}{2\sqrt{a^2-x^2}}\right)\right\}$$

$$=\frac{-x^2+a^2-x^2+a^2}{2\sqrt{a^2-x^2}}=\frac{a^2-x^2}{\sqrt{a^2-x^2}}=\sqrt{a^2-x^2}=\text{R.H.S}$$

OR

If $y=\log\left(x+\sqrt{x^2+1}\right)$, prove that $\left\{\left(x^2+1\right)\times\frac{d^2y}{dx^2}\right\}+x\left(\frac{dy}{dx}\right)=0$ [4]

Answer:

Given, $y=\log\left(x+\sqrt{x^2+1}\right)$

$$\Rightarrow\frac{dy}{dx}=\frac{1}{x+\sqrt{x^2+1}}\times\left(1+\frac{2x}{2\sqrt{x^2+1}}\right)=\frac{2\left(x+\sqrt{x^2+1}\right)}{\left(x+\sqrt{x^2+1}\right)\times2\left(\sqrt{x^2+1}\right)}=\frac{1}{\sqrt{x^2+1}}$$

Differentiating again, we get, $\frac{d^2y}{dx^2}=\left\{\frac{1}{2}\left(x^2+1\right)^{-\frac{3}{2}}\right\}\times2x=\frac{-x}{\left\{\left(x^2+1\right)^{\frac{3}{2}}\right\}}\Rightarrow\left(x^2+1\right)\times\frac{d^2y}{dx^2}=\frac{-x}{\sqrt{x^2+1}}$

Question: 17

Evaluate: $\int e^2x\sin x\,dx$

Answer:

$$\text{Let}, I = \int e^{2x} \sin x\, dx$$

$$= -e^{2x} \cos x + 2\int e^{2x} \cos x\, dx$$

$$= -e^{2x} \cos x + 2e^{2x} \sin x - \int 2e^{2x} \sin x\, dx$$

$$= -e^{2x} \cos x + 2e^{2x} \sin x - 4\int 2e^{2x} \sin x\, dx + C'$$

$$= -e^{2x}(2\sin x - \cos x) - 4I + C'$$

$$\Rightarrow I = \frac{e^{2x}}{5}(2\sin x - \cos x) + C\left(\text{where}, C = \frac{C'}{5}\right)$$

OR

Evaluate, $\int \frac{3x+5}{\sqrt{x^2-8x+7}}$

Answer:

Now, 3x 5 A $\frac{d}{dx}$ (x - x +) + B$\Rightarrow$ 3x + 5 =A (2x -8) + B $\Rightarrow$ 3x + 5 = 2Ax -8A + B

Equating the coefficient of x, and constant, we get

2A =3, and -8A + B =5 $A = \frac{3}{2}$, and $-8\, x\frac{3}{2} + B = 5$ $\Rightarrow$ B =5 +12 = 17

Hence,

$$\int \frac{3x+5}{\sqrt{x^2-8x+7}}dx = \left[\frac{\left\{\frac{3}{2}(2x-8)+17\right\}}{\left(\sqrt{x^2-8x+7}\right)}\right]dx = \frac{3}{2}\times\left\{\int \frac{(2x-8)}{\sqrt{x^2-8x+7}}\times dx\right\} + 17\int \frac{dx}{\sqrt{x^2-8x+7}} = \frac{3}{2}I_1 + I_2$$

$$\text{where}, I_1 = \int \frac{2x-8}{\sqrt{x^2-8x+7}}dx + I_2\int \frac{dx}{\sqrt{x^2-8x+7}}$$

$$\text{Now}, I_1 = \int \frac{2x-8}{\sqrt{x^2-8x+7}}dx$$

Let, x^2 -8x +7 = z 2 $\Rightarrow$ (2x -8) dx = 2zdz

$$I_1 = \int \frac{2zdz}{z} = 2\int dz = 2z + C_1 = 2\sqrt{x^2-8x+7} + C_1 \text{.............................(i)}$$

$$I_2 = \int \frac{dx}{\sqrt{x^2-8x+7}} = \int \frac{dx}{\sqrt{x^2-(2x\times 4)+16-16+7}} = \int \frac{dx}{\sqrt{(x-4)^2-3^2}}$$

$$= \log\left|(x-4)+\sqrt{(x-4)^2-3^2}\right| + C_2 = \log\left|(x-4)+\sqrt{x^2-8x+7}\right| + C_2$$

Putting the value of I_1 and I_2 in (i)

$$\int \frac{3x+5\, dx}{\sqrt{x^2-8x+7}} = \frac{3}{2}\left(2\sqrt{x^2-8x+7}\right) + 17\log\left|(x-4)+\sqrt{x^2-8x+7}\right| + (C_1 + C_2)$$

$$=3\sqrt{x^2-8x+7}+17\log\left|(x-4)\sqrt{x^2-8x+7}\right|+C \quad \left(\text{Note}: \int\frac{dx}{\sqrt{x^2-a^2}}=\log\left|x+\sqrt{x^2-a^2}\right|+C\right)$$

Question: 18 [4]

Find the particular solution of the differential equation: $(1+e^{2x})$ dy + $(1+y^2)$ = 0, given that y = 1, when x = 0.

Answer:

$$\left(1+e^{2x}\right)dy+\left(1+y^2\right)e^x dx=0 \Rightarrow \frac{dy}{1+y^2}+\frac{e^x dx}{1+e^{2x}}=0$$

Integrating both sides, we get:

$$\tan^{-1}y+\int\frac{e^x dx}{1+e^{2x}}=C\ldots\ldots\ldots.(1)$$

Let, $e^x=t\Rightarrow e^{2x}=t^2$

$$e^x=t\Rightarrow e^{2x}=t^2\Rightarrow e^x=\frac{dt}{dx}\Rightarrow e^x dx=dt$$

Substituting these values in equation (1), we get:

$$\tan^{-1}y+\int\frac{dt}{1+t_2}=C\Rightarrow\tan^{-1}y+\tan^{-1}t=C\Rightarrow\tan^{-1}y+\tan^{-1}\left(e^x\right)=C\ldots\ldots\ldots.(2)$$

Now, y = 1 at x = o.

Therefore, equation (2) becomes: $\tan^{-1}+\tan^{-1}1=C\Rightarrow\frac{\pi}{4}+\frac{\pi}{4}=C\Rightarrow C=\frac{\pi}{2}$

Substituting, $C=\frac{\pi}{2}$ in equation (2), we get, $\tan^{-1}y+\tan^{-1}\left(e^x\right)=\frac{\pi}{2}$

This is the required particular solution of the given differential equation.

Question: 19 [4]

Solve the following differential equation:

$\frac{dy}{dx}+y\cot x=4x\operatorname{cosec}x$, given that y = 0 when $x=\frac{\pi}{2}$

Answer:

Given differential equation is

$$\frac{dy}{dx}+y\cot x=4x\operatorname{cosec}x$$

$$\Rightarrow\frac{dy}{dx}+y\cot x.y=4x\operatorname{cosec}x$$

Comparing the given equation with $\frac{dy}{dx}+Py=Q$, we get

P = cot x, Q= 4x cosec x

$\therefore$ I.F $=e^{\int\cot x\,dx}$

$=e^{\log(\sin x)}=\sin x$

$y.\sin x=\int 4x.\operatorname{cosec}x.\sin x\,dx+C$

$\Rightarrow y.\sin x=\int 4x.\,dx+C$

$\Rightarrow y.\sin x = 2x^2 + C$

Putting y = 0 and $x = \frac{\pi}{2}$, we get

$0 = 2\frac{\pi^2}{2} + C \Rightarrow C = -\frac{\pi^2}{2}$

Therefore, required solution is $y \sin x = 2x^2 - \frac{\pi^2}{2}$

Question: 20 [4]

If vectors $\vec{a} = 2\hat{i} + 2\hat{j} + 3k$, $\vec{b} = \hat{i} + 2\hat{j} + k$ and $\vec{c} = 3\hat{i} + \hat{j}$ are such that $\vec{a} + \lambda\vec{b}$ is perpendicular to $\vec{c}$, then find the value of λ [4]

Answer:

Here $\vec{a} = 2\hat{i} + 2\hat{j} + 3k$, $\vec{b} = \hat{i} + 2\hat{j} + k$ and $\vec{c} = 3\hat{i} + \hat{j}$

$\vec{a} + \lambda\vec{b} = \left(2\hat{i} + 2\hat{j} + 3k\right) + \lambda\left(-\hat{i} + 2\hat{j} + k\right) = (2 - \lambda)\hat{i} + (2 + \lambda)\hat{j} + (3 + \lambda)k$

Since $\left(\vec{a} + \lambda\vec{b}\right)$ is perpendicular to $\vec{c}$

$\Rightarrow \left(\vec{a} + \lambda\vec{b}\right).\vec{c} = 0 \Rightarrow (2 - \lambda).3 + (2 - \lambda).1 + (3 + \lambda).0 = 0$

$\Rightarrow 6 - 3\lambda + 2 + 2\lambda = 0 \Rightarrow \lambda = 8$

Question: 21

Find the projection of $\vec{b} + \vec{c}$ on $\vec{a}$ where $\vec{a} = \left(2\hat{i} - 2\hat{j} + k\right)$, $\vec{b} = \left(\hat{i} + 2\hat{j} - 2k\right)$, and $\vec{c} = \left(2\hat{i} - \hat{j} + 4k\right)$.

Answer:

$\vec{b} + \vec{c} = \left(\hat{i} + 2\hat{j}\right) - \left(2k + 2\hat{i}\right) - \left(\hat{j} + 4k\right) = 3\hat{i} + \hat{j} + 2k$

Projection of $\left(\vec{b} + \vec{c}\right)$ on $\vec{a} = \frac{\left(\vec{b} + \vec{c}\right)\vec{a}}{\left|\vec{a}\right|} = \frac{\left\{\left(3\hat{i} + \hat{j} + 2k\right) \times \left(2\hat{i} - 2\hat{j}\right)\right\} + k}{\left(\sqrt{4 + 4 + 1}\right)} = \frac{(6 - 2 + 2)}{3} = \frac{6}{3} = 2$

Question: 22 [4]

Find the mean number of heads in three tosses of a fair coin.

Answer:

In any coin toss the $P(H) = P(T) = \frac{1}{2}$. A fair coin is tossed thrice. The sample space of three.

The sample space of three tosses of a coin is: $\begin{Bmatrix} HHH & HHTHTH & HTT \\ TTT & TTHTHT & THH \end{Bmatrix}$

If X is the random variable, we can see that X = 0, 1, 2 or 3 depending on the # of heads.

P (X = 0) = P (no heads) = P (TTT) $= \frac{1}{8}$

P (X = 1) = P (HTT, TTH and THT) $= \frac{3}{8}$

P (X = 2) = P (HHT, HTH and THH) $= \frac{3}{8}$

P (X = 3) = P (HHH) $= \frac{1}{8}$

Therefore, given this probability distribution, we can calculate the mean using the formula Mean of the probability distribution = $\sum\{X_i \times P(X_i)\}$

$$\Rightarrow \text{Mean} = \left\{\left(0\times\frac{1}{8}\right)+\left(1\times\frac{3}{8}\right)+\left(2\times\frac{3}{8}\right)+\left(3\times\frac{1}{8}\right)\right\} = \frac{12}{8} = \frac{3}{2}$$

Section: C

Question: 23 [6]

Use product $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}\begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$ to solve the system of equations:

x-y + 2z = 1
2y-3z =1
3x-2y+4z = 2

Answer:

Step: 1

$$\text{Let } A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & \sqrt{b^2 - 4ac}3 \\ 6 & 1 & -2 \end{bmatrix}$$

Now let us find the product AB, by the matrix multiplication.

$$AB = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}\begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix} = \begin{bmatrix} (-2-9+12) & 0-2+2 & 1+3-4 \\ (0+18-18) & 0+4-3 & 0-6+6 \\ (-6-18+24) & 0-4+4 & 3+6-8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Hence , } A^{-1} = B, \text{ i.e., } \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$$

Step 2:

$$\text{Now the system of equations of the form } AX = B, \text{ where, } A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix} X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\text{and, } A^{-1} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}, \text{therefore, } X = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2+0+2 \\ 9+2-6 \\ 6+1-4 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}$$

Hence, xd = 0, y = 5 and z = 3.

OR

Question: 24 [6]

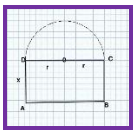
A window is in the form of a rectangle surrounded by a semi-circular opening. The total perimeter of the window is 10 meters. Find the dimensions of the rectangle so as to admit maximum light through the whole opening.

Answer:

Area of a circle = πr^2

Area of a rectangle = lb

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$



Step 1:

Perimeter of the window when the width of window is x, and 2r is the length.

$\Rightarrow 2x + 2r + \frac{1}{2} x\, 2\pi r = 10$

2x + 2r + π = 10, or,

2x + r(2+π) = 10 ……….(1)

For admitting the maximum light through the opening, the area of the window must be maximum.

A = Sum of areas of rectangle, and semi-circle.

Step: 2:

Area of circle = $\pi r^2 . \pi r^2$

Area of rectangle = l×b = 2×r×x

A = 2rx + $\left(\frac{1}{2} 2\pi + 2\right) r^2$

For maximum area $\frac{dA}{dr} = 0$, and $\frac{d^2A}{dr^2}$ is negative.

$\Rightarrow 10-(\pi+4)r = 0 \Rightarrow 10-(\pi+4)r=0$

or, $(\pi+4)r = 10(\pi+4)r=10$

or, r = $\frac{10}{\pi+4}$

Step 3:

$\frac{d^2A}{dr^2} = -(\pi+4) = -(\pi+4)$ (Differentiating with respect to r)

i.e., $\frac{d^2A}{dr^2}$ is negative for, $r = \frac{10}{\pi+4} \Rightarrow A \Rightarrow A$ is maximum.

From (1) we have $\Rightarrow 10 = (\pi+2)r + 2x \Rightarrow 10 = (\pi+2)r + 2x$

Put the value of rr in (1)

$$10 = (\pi+2) \times \left(\frac{10}{\pi+4}\right) + 2x = \left\{\frac{10(\pi+2)}{\pi+4}\right\} + 2x = \frac{10(\pi+2)+2x(\pi+4)}{(\pi+4)}$$

$10(\pi+4) = 10(\pi+2) + 2x(\pi+4)$
$10(\pi+4)-10(\pi+2)=2x(\pi+4)$
$10\pi+40-10\pi-20 = 2x(\pi+4)$
$20 = 2x(\pi+4)20$
$10=x(\pi+4)$

$$x = \frac{10}{\pi+4}$$

__Step:__ 4

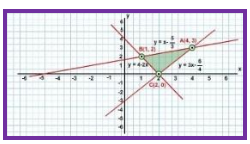
Length of rectangle, $2r = 2\left(\frac{10}{\pi+4}\right) = \frac{20}{\pi+4} \Rightarrow \frac{10}{\pi+4}$

Question: 25 [6]

Using the method of integration, find the area of the region bounded by the lines:
2x + y = 4
3x - 2y = 6
x - 3y + 5 = 0.

Answer:

The graph for the three lines 2x+y=4,3x-2y=6, and x-3y+5=0 can be sketched, and drawn as shown in the fig.



The required area is the region bounded by the lines in the area of the ΔABC.ΔABC. To find the limits, let us find the pints of intersection of the lines.

Let 2x+y = 4------(1)
3x-2y = 6------(2)
x-3y = -5------(3)

On solving (1), and (2) we get
(×3)(×3)2x+ y = 4
(×2)(×2)3x-2y = 6

6x+3y=12
-6x+3y=12

7y= 0
y= 0
x= 2

Hence, the point of intersection for line (1), and (2) is (2,0). On solving (2), and (3), we get

3x-2y= 6

(×3)(×3) x-3y =-5

3x-2y = 6
3x-9y = -15

7y = 21, or y = 3, x = 4. Hence the point of intersection between emu(2), and equation (3) is (4,3). On solving equation (3), and (1) we get,

(×2)(×2)x-3y=-5
2x+ y=4

2x-6y=-10
2x+ y= 4

-7y=-14
y=2.

Hence x = 1, and the point of intersection between emu(3), and equation (1) is (1,2).

The required area A
= (Area enclosed between the line AC, and x-axis) +
(Area enclosed between the line AB, and x-axis) +
(Area enclosed between the line BC, and x-axis)

$$A = \int_1^4 \left(\frac{x+5}{3}\right)dx + \int_2^1 (4-2x)dx + \int_4^2 \left(\frac{3x-6}{2}\right)dx$$

$$A = \left[\frac{1}{3} \times \left[\frac{x^2}{2} + 5x\right]_1^4\right] - \left[\frac{1}{3} \times \left[\frac{3x^2}{2} - 6x\right]_2^4\right]$$

$$A = \left[\frac{1}{3} \times \left[8 + 20 - \frac{1}{2}\right]_1^4\right] - [8-4-4]_1^2 - \left[\frac{1}{2} \times [24 - 24 - 6 + 12]_2^4\right]$$

$$A = \left[\left(\frac{1}{3} \times \frac{4}{5}\right) - \left\{1 - \frac{1}{2}\right\}\right](6)$$

$$A = \frac{15}{2} - 1 - 3 = \frac{7}{2} \text{ sq. units.}$$

Hence the required area is $\frac{7}{2}$ sq. units.

Question: 26 [6]

Find the equation of the plane passing through (1,2,3), and perpendicular to the straight line $\frac{x}{-2} = \frac{y}{-4} = \frac{z}{3}$

Answer:

Let Eq. of plane be Ax + By +Cz +D = 0, where A, B, and C are direction ratios of the normal to the plane. The straight line, $\frac{x}{-2} = \frac{y}{-4} = \frac{z}{3}$ is perpendicular to the plane, hence the line is normal to the plane equation of straight line passing through (x_1, y_1, z_1)

$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$ where a, b, and c are direction ratios of the line.

By comparison, we get A = -2, B = -4, and C = 3.
∴ Direction ratios of the normal to the plane are -2, -4, and 3.
∴ A = -2, B = -4 and C = 3

By placing the value of A, B, and C in the equation of plane, we get -2x -4x +3z +D =0

It passes though (1,2,3)
∴ -2(1)-4(2)+3(3)+D =0
-2 -8 +9 +D = 0
D-1=0 $\Rightarrow$ D=1

∴ Eq. of plane is -2x-4y+3z+1 = 0
or, 2x+4y+-3z-1 = 0

Question: 27 [6]

Find the equation of the plane passing through the point (-1, 3, 2), and perpendicular to each of the planes: x+2y+3z = 5, and 3x+3y +z = 0

Answer:

Step: 1

Let the equation of the plane passing through the point (−1, 3, 2)(−1,3,2) be
a(x+1)+b(y−3)+c(z−2) = 0a(x+1)+b(y−3)+c(z−2) = 0------(1)

Here a, b, ca, b, c are the direction ratios of normal to the plane. If two planes are ⊥⊥, then,
$a_1a_2+b_1b_2+c_1c_2 = 0$

Plane (1) is ⊥ to the plane x+2y+3z=5. The direction cosines are (1, 2, 3)(1,2,3). Therefore,
a.1+b.2+c.3 = 0a.1+b.2+c.3 = 0 ⇒ a+2b+3c = 0 ⇒ a+2b+3c=0------(2)

Step: 2

Also it is given plane (1) is ⊥ to 3x+3y+z=0 . The direction cosines are (3, 3,1)(3,3,1). Therefore,
a.3+b.3+c.1 = 0a.3+b.3+c.1 = 0 ⇒ 3a+3b+c = 0 ⇒ 3a+3b+c = 0------(3)

Step: 3

On solving equation (2), and equation (3) we get,

$$\frac{a}{\{(2\times1)-(3\times3)\}}=\frac{b}{(3\times3)-(1\times1)}=\frac{c}{\{(1\times3)-(2\times3)\}}\text{, i.e.,}\frac{a}{-7}=\frac{b}{8}=\frac{c}{-3}$$

Step: 4

Let this be equal to k, $\frac{a}{-7}=\frac{b}{8}=\frac{c}{-3}=k$

or, a = −7ka = −7k, b = 8kb = 8k, c = −3k

Step: 5

Substituting the value of a, ba, b, and cc in equation (1) we get
−7k(x+1)+8k(y−3)−3k(z−2) = 0−7k(x+1)+8k(y−3)−3k(z−2) = 0
−7kx−7k+8ky−24k−3kz+6k = 0−7kx−7k+8ky−24k−3kz+6k = 0
k[−7x−7+8y−24−3z+6] = 0k[−7x−7+8y−24−3z+6] = 0
⇒−7x−7+8y−24−3z+6=0⇒−7x−7+8y−24−3z+6 = 0

$\Rightarrow -7x+8y-3z-25 = 0 -7x+8y-3z-25 = 0$
$\Rightarrow 7x-8y+3z+25 = 0 7x-8y+3z+25 = 0$

Hence this is the required equation of the plane.

Question: 28

A cottage industry manufactures pedestal lamps, and wooden shades, each requiring the use of grinding/cutting machine, and a sprayer. It takes 2 hours on the grinding/cutting machine, and 3 hours on the sprayer to manufacture a pedestal lamp.

It takes one hour on the grinding/cutting machine, and 2 hours on the sprayer to manufacture a shade. On any day, the sprayer is available for at the most 20 hours, and the grinding/cutting machine for at the most 12 hours.

The profit from the sale of a lamps is □5, and that from a shade is □3. Assuming that the manufacturer can sell all the lamps, and shades that he produces, how should he schedule his daily production in order to maximize his profit? Make an L.P.P and solve it graphically.

Answer:

Let the cottage industry manufacture x pedestal lamps, and y wooden shades. Therefore, $x \geq 0$, and $y \geq 0$. The given information can be compiled in a table as follows,

	Pedestal Lamps (X)	Wooden Shades (Y)	Time Availability
Grinding/Cutting Machine(h)	2	1	12h
Sprayer (h)	3	2	20h
Profit (Rs.)	5	3	

We have the following constraints: $2x+y \leq\leq 12$, and $3x+2y \leq\leq 20$. The profit on pedestal lamps is ₹5, and on wooden lampshades is □3. We need to maximize the profits, i.e., maximize $5x + 3y$, given the above constraints.

Plotting the constraints:

Plot the straight lines, $2x + y = 12$, and $3x + 2y = 20$
First draw the graph of the line, $2x + y = 12$

If $x = 0$, $y = 12$, and if $y = 0$, $x = 6$. So, this is a straight line between (0, 12), and (6, 0). At (0, 0), in the inequality, we have $0 + 0 = 0$ which is $\leq\leq 0$. So the area associated with this inequality is bounded towards the origin.

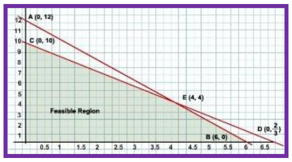
Similarly, draw the graph of the line $3x + 2y = 20$. If $x = 0$, $y = 10$, and if $y = 0$, $x = \frac{20}{3}$. So, this is a straight line between (0, 10), and $\left(\frac{20}{3}, 0\right)$. At (0, 0), in the inequality, we have $0 + 0 = 0$, which is $\leq\leq 0$. So the area associated with this inequality is bounded towards the origin.

Finding the feasible region finding the feasible region:

We can see that the feasible region is bounded, and in the first quadrant. On solving the equations $2x + y = 12$, and $3x + 2y = 20$, we get, $3x + 2(12-2x) = 20 \rightarrow\rightarrow 3x + 24 - 4x = 20 \rightarrow\rightarrow x = 4$.

If $x = 4$, $y = 12 - 2x4 = 4$ $\Rightarrow x=4, y=4 \Rightarrow x=4, y=4$

Therefore, the feasible region has the corner points (0, 0), (0, 10), (4, 4), (6, 0) as shown in the figure.



Solving the objective function using the corner point method. Solving the objective function using the corner point method. The values of Z at the corner points are calculated as follows:

Corner Point	*Z = 5x + 3y*
O(0,0)	0
C(0,0)	30
E(4,4)	32(Max value)
B (6,0)	30

The maximum profit we can make is □32, which involves making 4 pedestal lamps, and 4 wooden shades.

Question: 29

A factory has two machines A, and B. Past record shows that machine A produced 60% of the items of output, and machine B produced 40% of the items. Further, 2% of the items produced by machine A, and 1% produced by machine B were effective. All the items are put into one stockpile, and then one item is chosen at random form this, and is found to be defective. What is the probability that it was produced by machine B?

Answer:

$E_{1,}$ and E_2 are the events the percentage of production of items by machine A, and machine B respectively. Let A denotes defective item.

Machine A's production of items = 60%

$\therefore$ Probability of production of items by machine A, $P(E_1)$ = 60% = 0.9

Probability of production of items by machine A, $P(E_2)$ = 40% = 0.4

Probability that machine A produced defective item $P\left(\frac{A}{E_1}\right) =$ 2% = 0.02

Probability that machine B produced defective item $P\left(\frac{A}{E_2}\right) =$ 1% = 0.01

Thus, when, $P(E_1) = 0.6$, $P(E_2) = 0.4$, the, $P\left(\frac{A}{E_1}\right) = 0.02$, $P\left(\frac{A}{E_2}\right) = 0.01$

We have to find the probability that the defective item selected at random was from machine,

$$B = P\left(\frac{A}{E_2}\right) = P\left[\frac{P(E_2)P\left(\frac{A}{E_1}\right)}{\left\{P(E_1)P\left(\frac{A}{E_1}\right)\right\} + \left\{P(E_2)P\left(\frac{A}{E_2}\right)\right\}}\right] = \left[\frac{(0.4 \times 0.01)}{\{(0.6 \times 0.02) + (0.4 \times 0.01)\}}\right] = \frac{4}{16} = \frac{1}{4}$$