2008

Section A *(Question numbers 1 to 6 carry 1 mark each)*

Question: 1

Write the element a_{12} of the matrix A = $\left[a_{ij}\right]_{2\times2}$ whose elements a_{ij} are given by $a_{ij} = e^{2ix}\sin jx$.

Answer:
From given problem,
For a_{12} ,
i = 1
j = 2

$a_{ij} = e^{2ix}\sin jx$

$\Rightarrow a_{12} = e^{2.1x}\sin 2x \;\; = e^{2x}\sin 2x$

Question: 2
Find the differential equation of the family of lines passing through the origin.

Answer:
Equation of line passing through the origin is:

y – 0 = m(x – 0), or
$\Rightarrow$ $y = mx$, or

$$\frac{dy}{dx} = m \qquad (1)$$

Now, substituting, $m = \frac{dy}{dx}$ in equation (1) as,

y = mx, or

$y = \left(\frac{dy}{dx}\right)x$

Therefore, $y = \left(\frac{dy}{dx}\right)x$ is the required differential equation.

Question: 3

Find the integrating factor for the following differential equation: $x\log x\frac{dy}{dx} + y = 2\log x$

Answer:

We have given, $x\log x\frac{dy}{dx} + y = 2\log x$, or

$\left\{\left(\frac{x\log x}{x\log x}\right)\times\left(\frac{dy}{dx}\right)\right\} + \left(\frac{y}{x\log x}\right) = \left(\frac{2\log x}{x\log x}\right)$, or

$\frac{dy}{dx} + \frac{y}{x\log x} = \frac{2}{x}$

$p = \frac{1}{x\log x}$

I.F = $e^{\int P dx}$ = $e^{\int \frac{1}{x\log x} dx}$

Let, Log x = t

$\Rightarrow \frac{1}{x}dx = dt$

$\Rightarrow I.F = e^{\int \frac{dt}{t}} = e^{\log t} = t = \log x$

Question: 4

If $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$, and $\vec{b} = 3\hat{i} + 5\hat{j} - 2\hat{k}$. Then, find $|a \times b|$.

Answer:

From given value we get

$$a \times b = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 3 \\ 3 & 5 & -2 \end{vmatrix}$$

$= \hat{i}\{1.(-2) - 5.3\} - \hat{j}\{2.(-2) - 3.3\} + \hat{k}\{2.5 - 3.1\}$

$= \hat{i}\{-2 - 15\} - \hat{j}\{-4 - 9\} + \hat{k}\{10 - 3\}$

$= \hat{i}(-17) - \hat{j}(-13) + \hat{k}(7)$

$= 17\hat{i} + 13\hat{j} + 7\hat{k}$

$\Rightarrow |a \times b| = \sqrt{(17)^2 + (13)^2 + (7)^2} \quad = \sqrt{289 + 169 + 49} \quad = \sqrt{507}$

Question: 5

Find the angle between the vectors $\hat{i} - \hat{j}$, and $\hat{j} - \hat{k}$

Answer:

As we are have given

$\vec{a} = \hat{i} - \hat{j}$

$\vec{b} = \hat{j} - \hat{k}$

Then,

$\cos\theta = \dfrac{(\hat{i} - \hat{j}).(j - k)}{\sqrt{1^2 + 0^2 + 1^2}\sqrt{0^2 + 1^2 + 1^2}} \quad = \dfrac{-1}{\sqrt{2}\sqrt{2}} \quad = \cos\left(\pi - \dfrac{\pi}{3}\right) \quad = \cos\dfrac{2\pi}{3}$, or

$\theta = \dfrac{2\pi}{3}$

Question: 6

Find the distance of the point (2,5, - 3) from the plane $r.\left(6\hat{i} - 3\hat{j} + 2\hat{k}\right) = 0$

Answer:

$r.\left(6\hat{i} - 3\hat{j} + 2\hat{k}\right) = 0$

$\Rightarrow \left(x\hat{i} + y\hat{j} + z\hat{k}\right).\left(6\hat{i} - 3\hat{j} + 2\hat{k}\right) = 4$

$\Rightarrow 6x - 3y + 2k = 4$

Distance of point (2, 5, - 3) from 6x – 3y + 2k = 4 is calculated as,

$$d = \left|\frac{Ax_1 + By_1 + Cz_1 - 4}{\sqrt{6^2 + 3^2 + 2^2}}\right| = \left|\frac{6(2) - 3(5) + 2(-3) - 4}{\sqrt{6^2 + 3^2 + 2^2}}\right| = \left|\frac{12 - 15 - 6 - 4}{\sqrt{6^2 + 3^2 + 2^2}}\right| = \left|\frac{-13}{\sqrt{49}}\right| = \frac{13}{7}$$

Section: B *(Question numbers 7 to 19 carry 4 mark each)*

Question: 7

Evaluate : $\int \frac{x^2}{x^4 + x^2 - 2} dx$

Answer:

From given problem,

$$\Rightarrow \int \left(\frac{x^2}{x^4 + x^2 - 2} \right) dx$$

$$\Rightarrow \int \left\{ \frac{x^2}{x^4 + (2-1)x^2 - 2} \right\} dx$$

$$\Rightarrow \int \left(\frac{x^2}{x^4 + 2x^2 - x^2 - 2} \right) dx$$

$$\Rightarrow \int \left\{ \frac{x^2}{x^2(x^2+2) - 1(x^2+2)} \right\} dx$$

$$\Rightarrow \int \left\{ \frac{x^2}{(x^2-1)(x^2+2)} \right\} dx$$

Rewrite this as

$$\frac{x^2}{(x^2-1)(x^2+2)} = \frac{y}{(y-1)(y+2)}$$

By partial fraction.

$$\frac{y}{(y-1)(y+2)} = \left\{ \frac{A}{(y-1)} \right\} + \left\{ \frac{B}{(y+2)} \right\}$$

y = A(y + 2) + B (y – 1), or

y = Ay + 2A + By – B

A + B = 1
2A – B = 0

Solving these we get

3A = 1, or

$A = \frac{1}{3}$, and B = $\frac{2}{3}$

Therefore,

$$\frac{y}{(y-1)(y+2)} = \frac{1}{3(y-1)} + \frac{2}{3(y+2)} = \frac{1}{3} \times \left\{ \frac{1}{(y-1)} + \frac{2}{(y+2)} \right\}$$

Integrating as,

$$\int \frac{1}{3} \times \left\{ \frac{1}{(y-1)} + \frac{2}{(y+2)} \right\}$$

$$= \int \frac{1}{3} \times \left\{ \frac{1}{\left(x^2-1\right)} + \frac{2}{\left(x^2+2\right)} \right\} dx$$

$$= \frac{1}{3} \times \left[\left\{ \frac{1}{2} \times \left(\frac{\log|x-1|}{\log|x+1|} \right) \right\} + \left\{ \frac{2}{\sqrt{2}} \times \tan^{-1}\left(\frac{x}{\sqrt{2}} \right) \right\} \right]$$

$$= \left\{ \frac{1}{6} \times \left(\frac{\log|x-1|}{\log|x+1|} \right) \right\} + \left\{ \sqrt{2} \tan^{-1}\left(\frac{x}{\sqrt{2}} \right) \right\}$$

Question: 8

In a set of 10 coins, 2 coins are with heads on the both sides. A coin is selected at random from the set and tossed five times. If all the five, the result was heads, find the probability that the selected coin had heads on the both sides.

Answer:

Let E_1 be the event of selected two head coin, E_2 be the event of selecting one headed coin, and H be the event of getting head.

Then,

$P(E_1) = \frac{2}{10} = \frac{1}{5}$, and

$P(E_2) = \frac{8}{10} = \frac{4}{5}$

Since on two headed coin probability of getting head is 1 as head exists on both sides. Therefore, if the coin is tossed 5 times then the probability of getting head is

$$P\left(\frac{E_1}{H}\right) = 1^5$$

On an unbiased coin, the probability of getting head is $\left(\frac{1}{2}\right)$, therefore, when the coin is tossed five times, the probability of getting head is,

$$P\left(\frac{E_2}{H}\right) = \left(\frac{1}{2}\right)^5$$

The probability that the selected coin had heads on both the sides is:

$$P\left(\frac{H}{E_1}\right) = \left[\frac{\left\{ P(E_1) \times P\left(\frac{E_1}{H}\right) \right\}}{\left\{ P(E_1) \times P\left(\frac{E_1}{H}\right) \right\} + \left\{ P(E_2) \times P\left(\frac{E_2}{H}\right) \right\}} \right] = \left[\frac{\left(\frac{1}{5} \times 1\right)}{\left(\frac{1}{5} \times 1\right) + \left\{ \frac{4}{5} \times \left(\frac{1}{2}\right)^5 \right\}} \right]$$

$$= \frac{\frac{1}{5}}{\frac{1}{5}+\left(\frac{4}{5}\times\frac{1}{32}\right)} = \frac{1}{1+\frac{1}{8}} = \frac{1}{\frac{9}{8}} = \frac{8}{9}$$

Hence, the probability that the selected coin has head on both sides is $P\left(\frac{H}{E_1}\right)=\frac{8}{9}$

OR

How many times must a fair coin be tossed so that the probability of at least one head is more than 80%?

Answer:

We are given, the probability of getting at least one head is 80%. This implies the value of head can be 1 or 2 but not 0. This implies,

$$P(X \geq 1) > 80\% \; > \frac{80}{100}\text{, or}$$

$$1-P(X=0) > \frac{80}{100}\text{, or}$$

$$P(X=0) < 1-\left(\frac{80}{100}\right)\text{, or}$$

$$P(X=0) < \left(\frac{2}{10}\right)\text{, or}$$

$$^{n}C_0\left(\frac{1}{2}\right)^n < \frac{1}{5}\text{, or}$$

$$\left(\frac{1}{2}\right)^n < \frac{1}{5}$$

To make the denominator bigger the value of n would be more than 2 as with n = 1, and 2 we get a denominator as 2, 4 which is less than 5 so not satisfying the inequality.

To satisfy this inequality the value can be n = 3, 4, 5 …..

Hence, man must toss at least 3 times.

Question: 9

Find x, such that the four points A(4,1, 2), B(5, x, 6), C(5,1, -1), and D(7, 4, 0) are coplanar.

Answer:

From given four points let us find,

$$\overrightarrow{AB} = (5-4, x-1, 6-2)$$

$$\overrightarrow{AC} = (5-4, 1-1, -1-2)$$

$$\overrightarrow{AD} = (7-4, 4-1, 0-2)$$

Therefore,

$\overrightarrow{AB} = (1, x-1, 4)$

$\overrightarrow{AC} = (1, 0, -3)$

$\overrightarrow{AD} = (3, 3, -2)$

For vectors to be complanar,

$\overrightarrow{AB}.\left(\overrightarrow{AC} \times \overrightarrow{AD}\right) = 0$

$$\Rightarrow \begin{vmatrix} 1 & x-1 & 4 \\ 1 & 0 & -3 \\ 3 & 3 & -2 \end{vmatrix} = 0$$

$\Rightarrow$ 1(0 + 9) – (x - 1) (-2 + 9) + 4(3 – 0) = 0

$\Rightarrow 9 - 7(x-1) + 12 = 0$

$\Rightarrow 9 - 7x - 7 + 12 = 0$

$\Rightarrow 14 - 7x = 0$

$\Rightarrow 7x = 14$

$\Rightarrow x = 2$

Question: 10

A line passing through the point A with position vector $\vec{a} = 4\hat{i} + 2\hat{j} + 2\hat{k}$ is parallel to the vector $\vec{b} = 2\hat{i} + 2\hat{j} + 6\hat{k}$. Find the length of the perpendicular drawn on this line from a point P with position vector $r_1 = \hat{i} + 2\hat{j} + 3\hat{k}$.

Answer:

We are given vector $\vec{a} = 4\hat{i} + 2\hat{j} + 2\hat{k}$, and vector $\vec{b} = 2\hat{i} + 2\hat{j} + 6\hat{k}$. The equation of a line passing through a point $\vec{a} = 4\hat{i} + 2\hat{j} + 2\hat{k}$, and parallel to vector $\vec{b} = 2\hat{i} + 2\hat{j} + 6\hat{k}$ is given by:

$\vec{r} = \vec{a} + \lambda\vec{b}$, or

$\vec{r} = \left(4\hat{i} + 2\hat{j} + 2\hat{k}\right) + \lambda\left(2\hat{i} + 2\hat{j} + 6\hat{k}\right)$

Rewrite this equation in Cartesian form as, $\dfrac{x-4}{2} = \dfrac{y-2}{2} = \dfrac{z-2}{6} = \lambda$

This implies

$x = 2\lambda + 4$

$y = 2\lambda + 2$

$z = 6\lambda + 2$

Since the line is parallel to the line having normal vector $\vec{b} = 2\hat{i} + 2\hat{j} + 6\hat{k}$

Therefore,

$2(2\lambda + 4) + 2(2\lambda + 2) + 6(6\lambda + 2) = 0$

$\Rightarrow 4\lambda + 8 + 4\lambda + 4 + 36\lambda + 12 = 0$

$\Rightarrow 44\lambda + 24 = 0$

$\Rightarrow \lambda = -\dfrac{24}{44}$

$$\Rightarrow \lambda = -\frac{6}{11}$$

$$x = 2\lambda + 4 \quad = 2\left(-\frac{6}{11}\right) + 4 \quad = \frac{20}{11}$$

$$y = 2\lambda + 2 \quad = 2\left(-\frac{6}{11}\right) + 2 \quad = \frac{10}{11}$$

$$z = 6\lambda + 2 \quad = 6\left(-\frac{6}{11}\right) + 2 \quad = -\frac{14}{11}$$

So the point is $\left(\frac{20}{11}, \frac{10}{11}, \frac{14}{11}\right)$

Distance between $\left(\frac{20}{11}, \frac{10}{11}, \frac{14}{11}\right)$, and $(4,2,2)$ is given by,

$$d = \sqrt{\left(\frac{20}{11} - 4\right)^2 + \left(\frac{10}{11} - 2\right)^2 + \left(-\frac{14}{11} - 2\right)^2} \quad = \sqrt{\left(\frac{576 + 144 + 1296}{121}\right)} \quad = \sqrt{16 \times \left(\frac{80}{121}\right)} = 4.08$$

Question: 11

Solve the following for x : $\sin^{-1}(1-x) - 2\sin^{-1}x = \frac{\pi}{2}$

Answer:

We are given, $\sin^{-1}(1-x) - 2\sin^{-1}x = \frac{\pi}{2}$. This can be simplified as,

$$\sin^{-1}(1-x) - 2\sin^{-1}x = \frac{\pi}{2}$$
$$\Rightarrow -2\sin^{-1}x = \frac{\pi}{2} - \sin^{-1}(1-x)$$
$$\Rightarrow -2\sin^{-1}x = \cos^{-1}(1-x)$$

Substitute, $x = \sin\theta$ as,

$$-2\sin^{-1}x = \cos^{-1}(1-x)$$
$$\Rightarrow -2\sin^{-1}(\sin\theta) = \cos^{-1}(1-\sin\theta)$$
$$\Rightarrow -2\theta = \cos^{-1}(1-\sin\theta)$$
$$\Rightarrow \cos(-2\theta) = 1-\sin\theta$$
$$\Rightarrow \cos 2\theta = 1-\sin\theta$$
$$\Rightarrow 1-2\sin^2\theta = 1-\sin\theta$$
$$\Rightarrow 2\sin^2\theta - \sin\theta = 0$$
$$\Rightarrow \sin\theta(2\sin\theta - 1) = 0$$

Either $\sin\theta = 0$ or $(2\sin\theta - 1) = 0$

$\therefore \theta = 0$ or 30^0

$\because x = \sin\theta$ (given) $= 0, \frac{1}{2}$

OR

Question:

Show that: $2\sin^{-1}\frac{3}{5} - \tan^{-1}\left(\frac{17}{31}\right) = \frac{\pi}{4}$

Answer:

From L.H.S

$$2\tan^{-1}\frac{3}{4} - \tan^{-1}\left(\frac{17}{31}\right) = \tan^{-1}\left(\frac{2.\frac{3}{4}}{1-\left(\frac{3}{4}\right)^2}\right) - \tan^{-1}\left(\frac{17}{31}\right) = \tan^{-1}\left(\frac{\frac{3}{2}}{\frac{16-9}{16}}\right) - \tan^{-1}\left(\frac{17}{31}\right)$$

$$= \tan^{-1}\left(\frac{\frac{3}{2}}{\frac{7}{16}}\right) - \tan^{-1}\left(\frac{17}{31}\right) = \tan^{-1}\left(\frac{24}{7}\right) - \tan^{-1}\left(\frac{17}{31}\right) = \tan^{-1}\left(\frac{\frac{24}{7}-\frac{17}{31}}{1+\frac{24}{7}\times\frac{17}{31}}\right) = \tan^{-1}\left(\frac{744-119}{217+408}\right)$$

$$= \tan^{-1}\left(\frac{625}{625}\right) = \tan^{-1}(1) = \frac{\pi}{4} \text{ (R.H.S)}$$

Question: 12

If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$. Then show that $A^2 - 4A - 5I = 0$, and hence find A^{-1}

Answer:

Taking L.H.S as, $A^2 - 4A - 5I$

$$= \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - 5\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

To find A^{-1}, multiply both sides by A^{-1} as,

$A^2 - 4A - 5I$

$\Rightarrow A \times \left(AA^{-1}\right) - 4AA^{-1} - 5IA^{-1} = 0A^{-1}$

$\Rightarrow AI - 4I - 5A^{-1} = 0$

$\Rightarrow A - 4I - 5A^{-1} = 0$

$\Rightarrow A - 4I = 5A^{-1}$

Substituting the value of A, and I,

$$A^{-1}=\frac{1}{5}(A-4I)=\frac{1}{5}\times\left\{\begin{pmatrix}1&2&2\\2&1&2\\2&2&1\end{pmatrix}-4\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right\}=\frac{1}{5}\times\left\{\begin{pmatrix}1&2&2\\2&1&2\\2&2&1\end{pmatrix}-4\begin{pmatrix}4&0&0\\0&4&0\\0&0&4\end{pmatrix}\right\}$$

$$=\frac{1}{5}\times\begin{pmatrix}-3&2&2\\2&-3&2\\2&2&-3\end{pmatrix}=\begin{pmatrix}-\frac{3}{5}&\frac{2}{5}&\frac{2}{5}\\\frac{2}{5}&-\frac{3}{5}&\frac{2}{5}\\\frac{2}{5}&\frac{2}{5}&-\frac{3}{5}\end{pmatrix}$$

OR

If $A=\begin{bmatrix}2&0&-1\\5&1&0\\0&1&3\end{bmatrix}$ find A^{-1} by elementary row transformation.

Answer:

Rewriting the given problem as, $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}A$

Applying, $R_1\to\frac{R_1}{2}$: $\begin{bmatrix}1&0&-\frac{1}{2}\\5&1&0\\0&1&3\end{bmatrix}=\begin{bmatrix}\frac{1}{2}&0&0\\0&1&0\\0&0&1\end{bmatrix}A$

Applying, $R_2\to R_3-5R_1$: $\begin{bmatrix}1&0&-\frac{1}{2}\\0&1&\frac{5}{2}\\0&0&3\end{bmatrix}=\begin{bmatrix}\frac{1}{2}&0&0\\-\frac{5}{2}&1&0\\\frac{5}{2}&-1&1\end{bmatrix}A$

Applying, $R_2\to R_3-R_2$: $\begin{bmatrix}1&0&-\frac{1}{2}\\0&1&\frac{5}{2}\\0&0&\frac{1}{2}\end{bmatrix}=\begin{bmatrix}\frac{1}{2}&0&0\\-\frac{5}{2}&1&0\\\frac{5}{2}&-1&1\end{bmatrix}A$

Applying, $R_1\to R_1+\frac{1}{2}R_3$

Applying, $R_2 \to R_2 - \frac{5}{2}R_3$: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} A$

$$\Rightarrow A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

Question: 13

Using the properties of determinants, solve the following for x.

$$\begin{vmatrix} x+2 & x+6 & x-1 \\ x+6 & x-1 & x+2 \\ x-1 & x+2 & x+6 \end{vmatrix} = 0$$

Answer:

From given problem we can be presented as $\begin{vmatrix} 3x+7 & 3x+7 & 3x+7 \\ x+6 & x-1 & x+2 \\ x-1 & x+2 & x+6 \end{vmatrix} = 0$

Applying, $R_1 \to R_1 + R_2 + R_3$

$$\Rightarrow (3x+7) \times \begin{vmatrix} 3x+7 & 3x+7 & 3x+7 \\ x+6 & x-1 & x+2 \\ x-1 & x+2 & x+6 \end{vmatrix} = 0$$

$$\Rightarrow (3x+7) \times \begin{vmatrix} 1 & 0 & 0 \\ x+6 & -7 & 4 \\ x-1 & 3 & 7 \end{vmatrix} = 0$$

Applying, $C_2 \to C_2 - C_1$, and $C_3 \to C_3 - C_1$

Expanding along R_1

$$\Rightarrow (3x+7)\{1(-7.7-3.4)\} = 0$$
$$\Rightarrow (3x+7)(-49-12) = 0$$
$$\Rightarrow (3x+7)(-61) = 0$$
$$\Rightarrow (3x+7) = 0$$
$$\Rightarrow x = -\frac{7}{3}$$

Question: 14

Evaluate: $I=\int_0^{\frac{\pi}{2}}\left(\frac{\sin^2 x}{\sin x+\cos x}\right)dx$

Answer:

$$I=\int_0^{\frac{\pi}{2}}\frac{\sin^2 x}{\sin x+\cos x}dx\ =\int_0^{\frac{\pi}{2}}\left\{\frac{\sin^2\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}\right\}dx\ =\int_0^{\frac{\pi}{2}}\left(\frac{\cos^2 x}{\cos x+\sin x}\right)dx$$

$$2I=\int_0^{\frac{\pi}{2}}\frac{\sin^2 x+\cos^2 x}{\cos x+\sin x}dx\ =\int_0^{\frac{\pi}{2}}\left(\frac{1}{\cos x+\sin x}\right)dx$$

$$\Rightarrow I=\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{1}{\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)+\left(\frac{2\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}dx$$

$$\Rightarrow I=\frac{I}{2}\int_0^{\frac{\pi}{2}}\left(\frac{1+\tan^2\frac{x}{2}}{1-\tan^2\frac{x}{2}+2\tan^2\frac{x}{2}}\right)dx$$

$$\Rightarrow I=\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{\sec^2\frac{x}{2}}{1-\left(\tan^2\frac{x}{2}-2\tan^2\frac{x}{2}\right)}dx$$

$$\Rightarrow I=\frac{1}{2}\int_0^{\frac{\pi}{2}}\left\{\frac{\sec^2\frac{x}{2}}{1-\left(\tan^2\frac{x}{2}-2\tan^2\frac{x}{2}+1-1\right)}\right\}dx$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\left\{\frac{\sec^2\frac{x}{2}}{2-\left(\tan^2\frac{x}{2}-1\right)^2}\right\}dx$$

Let, $\tan\frac{x}{2}-1=t$

$\frac{1}{2}\sec^2\frac{x}{2}\,dx=dt$

Also, if

$\tan\frac{x}{2} - 1 = t$

$\tan 0 - 1 = t$

$-1 = t$

And

$\tan\frac{\pi}{2} - 1 = t$, or

$\infty - 1 = t$, or

$t = \infty$

Therefore,

$$I = \frac{1}{2}\times\left[\frac{1}{2}\left\{\int_{-1}^{\infty}\left(\frac{dt}{2-t^2}\right)\right\}\right] = \frac{1}{4}\times\left\{\frac{1}{2\sqrt{2}}\left(\log\left|\frac{\sqrt{2}+x}{\sqrt{2}-x}\right|\right)\right\} = \frac{1}{4}\times\left[\frac{1}{2\sqrt{2}}\left\{\log\left|\frac{\sqrt{2}+\infty}{\sqrt{2}-\infty}\right| - \log\left|\frac{\sqrt{2}+(-1)}{\sqrt{2}-(-1)}\right|\right\}\right]$$

$$= \frac{1}{8\sqrt{2}}\left[0 - \log\left|\frac{\sqrt{2}-1}{\sqrt{2}+1}\right|\right] = \frac{1}{8\sqrt{2}}\left(\log\left|\frac{\sqrt{2}-1}{\sqrt{2}+1}\right|\right)$$

OR

Evaluate $\int_{-1}^{2}\left(e^{3x}+7x-5\right)$ as a limit of sums.

Answer:

From given problem, we get

$h = \frac{b-a}{n} = \frac{2+1}{n} = \frac{3}{n}$

Using formula,

$\lim_{h\to 0}\left[h\{f(a)+f(a+h)+f(a+2h)+.........\}\right]$

$\int_{-1}^{2}\left(e^{3x}+7x-5\right)$

$$= \lim_{h\to 0}\left[h\{f(a)+f(a+h)+f(a+2h)+...........\}\right]$$

$$= \lim_{h\to 0}\left[h\left\{\left(e^{-3}-7-5\right)+\left(e^{-3+h}-7+7h-5\right)+\left(e^{-3+2h}-7+14h-5\right)+...........\right\}\right]$$

$$= \lim_{h\to 0}\left[h\left\{e^{-3}\left(1+e^{h}+e^{2h}+e^{3h}.......\right)+7h\left(1+2+3.......\right)-12n\right\}\right]$$

$$= \lim_{\substack{n\to\infty\\h\to 0}}\left[\frac{3}{n}\left\{e^{-3}\left(1+e^{h}+e^{2h}+e^{3h}.........\right)+7h\left(1+2+3........\right)-12n\right\}\right]$$

$$= \lim_{\substack{n\to\infty\\h\to 0}}\frac{3}{n}\left[e^{-3}\left\{3\times\frac{1\left(e^{nh}-1\right)}{e^{h}-1}\right\}+7h\left\{\frac{n(n-1)}{2}\right\}-12n\right]$$

$$= \lim_{\substack{n\to\infty \\ h\to 0}} \frac{3}{n}\left[e^{-3}\left\{3\frac{1\left(e^{3}-1\right)}{3h\left(\frac{e^{h}-1}{h}\right)}\right\} + 7h\left\{\frac{n^{2}\left(1-\frac{1}{n}\right)}{2}\right\} - 12n \right]$$

$$= \lim_{\substack{n\to\infty \\ h\to 0}} \frac{3}{n}\left[\frac{3}{n}e^{-3}\left(\frac{3}{\frac{3}{n}}\right) + 7\left\{\frac{\left(\frac{3}{n}\right)^{2} n^{2}\left(1-\frac{1}{n}\right)}{2}\right\} - 12n.\frac{3}{n} \right]$$

$$= 3e^{-3} + \frac{63}{2} - 36$$

$$= \frac{6e^{-3} - 9}{2}$$

Question: 15

If $x = a\sin 2t(1+\cos 2t)$, and $x = a\sin 2t(1+\cos 2t)$, then find $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$.

Answer:

Differentiating both given equation with respect to t as,

$x = a\sin 2t(1+\cos 2t)$, or

$$\frac{dx}{dt} = 2a\cos 2t(1+\cos 2t) + a\sin 2t.(-2\sin 2t)$$

$$= 2a\cos 2t + 2a\cos^{2} 2t - 2a\sin^{2} 2t$$

$$= 2a\cos 2t + 2a\cos 4t$$

$$= 2a(\cos 4t + \cos 2t)$$

$y = b\cos 2t(1-\cos 2t)$, or

$$\frac{dy}{dt} = -2b\sin 2t(1-\cos 2t) + b\cos 2t(2\sin 2t)$$

$$= -2b\sin 2t + 2b\sin 2t\cos 2t + 2b\sin 2t\cos 2t$$

$$= -2b\sin 2t + 4b\sin 2t\cos 2t$$

$$= -2b\sin 2t + 2b\sin 4t$$

$$= 2b(\sin 4t - \sin 2t)$$

Therefore,

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{2b(\sin 4t - \sin 2t)}{2a(\cos 4t + \cos 2t)} = \frac{b(2\cos 3t\sin t)}{a(2\cos 3t\cos t)} = \frac{b}{a}\tan t$$

At $t = \frac{\pi}{4}$

$$\frac{dy}{dx}=\frac{b}{a}\tan t \;=\left(\frac{b}{a}\right)\tan\frac{\pi}{4}\;=\left(\frac{b}{a}\right)1\;=\frac{b}{a}$$

Hence, $\frac{dy}{dx}=\frac{b}{a}$

Question: 16

Evaluate: $\int\left\{\frac{(x+3)e^x}{(x+5)^3}\right\}dx$

Answer:

Analyzing given problem,

$$\int\left\{\frac{(x+3+2-2)e^x}{(x+5)^3}\right\}dx=\int\left\{\frac{(x+5-2)e^x}{(x+5)^3}\right\}dx\;=\int e^x\left\{\frac{(x+5)}{(x+5)^3}-\frac{2}{(x+5)^3}\right\}dx=\int e^x\left\{\frac{1}{(x+5)^2}-\frac{2}{(x+5)^3}\right\}dx$$

Using property: $\Rightarrow \int e^x\left(f(x)+f'(x)\right)=e^x f(x)$

$$\therefore \int e^x\left[\left\{\frac{1}{(x+5)^2}\right\}-\left\{\frac{2}{(x+5)^3}\right\}\right]dx=e^x\left\{\frac{1}{(x+5)^2}\right\}$$

Question: 17

Three schools X, Y, and Z organized a fete for collecting funds for flood victims in which they sold handheld fans, mats and toys made from recycled material, the sale Price of each being ₹25, ₹100, and ₹50 respectively. The following table shows the number of articles of each type sold.

Articles \ Schools	X	Y	Z
Hand held Fans	30	40	35
Mats	12	15	20
Toys	70	55	75

Using matrices, find the fund collected by each school by selling the above articles, and the total funds collected. Also write any one value generated by the above situation.

Answer:

The fund is generated by each school can be collected by matrix multiplication as :

$$\begin{bmatrix}25 & 100 & 50\end{bmatrix}\begin{bmatrix}30 & 40 & 35\\ 12 & 15 & 20\\ 70 & 55 & 75\end{bmatrix}$$

$$=\left[\{(25\times30)+(100\times12)+(50\times70)\}\quad\{25.40+(100\times15)+(50\times55)\}\quad\{(25\times35)+(100\times20)+(50\times75)\}\right]$$

$$=\left[(750+1200+3500)\quad(100+1500+2750)\quad(875+2000+3750)\right]$$

$= [5450 \quad 5250 \quad 6625]$

Hence, funds generated by school X, Y, and Z are ₹5450, ₹5250 , and ₹6625 respectively.

Question: 18

If $y = e^{ax}\cos bx$, then prove that, $\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (b^2 + a^2)y = 0$

Answer:

Differentiating w.r.t x twice as,

$y = e^{ax}\cos bx$

$\frac{dy}{dx} = ae^{ax}\cos bx - be^{ax}\sin bx = a^2e^{ax}\cos bx - abe^{ax}\sin bx - bae^{ax}\sin bx - b^2e^{ax}\cos bx$

$\Rightarrow \frac{d^2y}{dx^2} = (a^2 - b^2)e^{ax}\cos bx - 2abe^{ax}\sin bx$

To prove : $\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (b^2 + a^2)y = 0$

Using L.H.S, and substituting all Values as,

$\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (b^2 + a^2)y$

$= (a^2 - b^2)e^{ax}\cos bx - 2abe^{ax}\sin bx - 2a(ae^{ax}\cos bx - be^{ax}\sin bx) + (b^2 + a^2)e^{ax}\cos bx$

$= 2a^2e^{ax}\cos bx - 2abe^{ax}\sin bx - 2a^2e^{ax}\cos bx + 2abe^{ax}\sin bx$

= 0 (R.H.S)

Question: 19

If $x^x + x^y + y^x = a^b$, then find $\frac{dy}{dx}$

Answer:

$x^x + x^y + y^x = a^b$

$\Rightarrow u + v + w = a^b$

$\Rightarrow \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} = 0$

Then

$u = x^x$

Log u = x log x

$\left\{\left(\frac{1}{u}\right) \times \left(\frac{du}{dx}\right)\right\} = 1.\log x + x.\left(\frac{1}{x}\right) \quad = \log x + 1$

$\frac{du}{dx} = u(\log x + 1)$

$$\frac{du}{dx} = x^x(\log x + 1)$$

And

$v = x^y$

$\log v = y\log x$

$$\left\{\left(\frac{1}{v}\right)\times\left(\frac{dv}{dx}\right)\right\} = \left(\frac{dy}{dx}\right).\log x + \frac{y}{x}$$

$$\frac{dv}{dx} = x^y\left(\frac{dy}{dx}.\log x + \frac{y}{x}\right)$$

And

$w = y^x$

$\log v = y\log x$

$$\left\{\left(\frac{1}{v}\right)\times\left(\frac{dv}{dx}\right)\right\} = \frac{dy}{dx}\times\log x + \frac{y}{x}$$

$$\frac{dv}{dx} = x^y\left(\frac{dy}{dx}.\log x + \frac{y}{x}\right)$$

Therefore,

$$\frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} = 0$$

$$\Rightarrow x^x(\log x + 1) + x^y\left(\frac{dy}{dx}.\log x + \frac{y}{x}\right) + y^x\left(\log y + \frac{x}{y}.\frac{dy}{dx}\right) = 0$$

$$\Rightarrow \frac{dy}{dx}\left\{x^y\log x + y^x\left(\frac{x}{y}\right)\right\} = -x^x(\log x + 1) - x^y\left(\frac{y}{x}\right) - y^x\log y$$

$$\Rightarrow \frac{dy}{dx} = -\left\{\frac{x^x(\log x + 1) + x^y\frac{y}{x} + y^x\log y}{\left(x^y\log x + y^{x-1}x\right)}\right\}$$

Section C *(Question numbers 20 to 26 carry 6 mark each)*

Question: 20

Find the particular solution of the differential equation $x^2dy = (2xy + y^2)dx$ given that y = 1when x =1.

Answer:

From given equation,

$$\frac{dy}{dx} = \frac{\left(2xy + y^2\right)dx}{x^2} = \frac{2y}{x} + \frac{y^2}{x^2}$$

Substituting,

$\frac{y}{x} = v$, or

y = xv, or

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Therefore,

$$\Rightarrow v + x\frac{dv}{dx} = 2v + v^2$$

$$\Rightarrow x\left(\frac{dv}{dx}\right) = v + v^2$$

$$\Rightarrow \frac{dv}{v + v^2} = \frac{dx}{x}$$

$$\Rightarrow \frac{dv}{v\left(1+v\right)} = \frac{dx}{x}$$

By partial fraction,

$\frac{1}{v\left(1+v\right)} = \frac{A}{v} + \frac{B}{\left(1+v\right)} = A\left(1+v\right) + Bv$, or

1 = A(1+v) + Bv = A + Av + Bv, or

A+B = 0, or

A = 1, and B = -1

Therefore,

$$\frac{dv}{v\left(1+v\right)} = \frac{dx}{x}$$

$$\left\{\left(\frac{1}{v}\right) - \left(\frac{1}{1+v}\right)\right\} = \frac{dx}{x}$$

Integrating both sides as,

$\int\left(\frac{1}{v} - \frac{1}{1+v}\right)\int\frac{dx}{x}$, or

logv – log(1+v) = logx + logc

$$\Rightarrow \log\frac{v}{v+1}\log(xc)$$

$$\Rightarrow \frac{v}{v+1} = xc$$

Substituting, x = 1, and y = 1 as,

$$\Rightarrow \frac{\frac{y}{x}}{\frac{y}{x}+1} = xc$$

$$\Rightarrow \frac{\frac{1}{1}}{\frac{1}{1}+1}$$

$\Rightarrow$ 1 = c

Substitute the value of c in the above equation, therefore, the required equation is:

$$\frac{\frac{y}{x}}{\frac{y}{x}+1} = x(1)$$

$$\frac{y}{y+x} = x$$

y = x (x+y) = x^2 + xy, or
y – xy = x^2
y(1-x) = x^2

$$y = \frac{x^2}{(1-x)}$$

OR

Find the particular solution of the differential equation $x^2\,dy = \left(2xy + y^2\right)dx$ given that y = 1 when x = 0.

Answer:
Rewrite the given equation as,

$$\left(1+x^2\right)\frac{dy}{dx} = e^{m\tan^{-1}x} - y \text{ , or}$$

$$\frac{dy}{dx} = \frac{e^{m\tan^{-1}x} - y}{\left(1+x^2\right)} = \frac{e^{m\tan^{-1}x}}{\left(1+x^2\right)} - \frac{y}{\left(1+x^2\right)} \text{ , or}$$

$$\frac{dy}{dx} + \frac{y}{\left(1+x^2\right)} = \frac{e^{m\tan^{-1}x}}{\left(1+x^2\right)}$$

This is a linear equation.

Its Solution is given by, $y.I.F = \int Q.I.F + C$

Here,

$$P = \frac{1}{\left(1+x^2\right)};\ Q = \frac{e^{m\tan^{-1}x}}{1+x^2}$$

$$I.F = e^{\int p dx} = e^{\int \frac{1}{(1+x^2)}dx} = e^{\tan^{-1}x}$$

$$y.e^{\tan^{-1}x} = \left(\frac{1}{m+1}\right)\int e^t dt + C = \left(\frac{1}{m+1}\right)e^t + C$$

At x = 0 ; y = 1, Substituting in the above equation,

$$y = \frac{1}{m+1}e^{m\tan^{-1}x} + Ce^{-\tan^{-1}x} \text{ , or}$$

$$1 = \frac{1}{m+1}e^{m\tan^{-1}0} + Ce^{-\tan^{-1}0}$$

$$1 = \frac{1}{m+1} + C$$

$$C = 1 - \frac{1}{m+1} = \frac{m+1-1}{m+1} = \frac{m}{m+1}$$

Substitute the value of c in above equation as,

$$y = \frac{1}{m+1}e^{m\tan^{-1}x} + Ce^{-\tan^{-1}x} = \frac{1}{m+1}e^{m\tan^{-1}x} + \frac{m}{m+1}e^{-\tan^{-1}x}$$

This is the required equation.

Question: 21

Find the absolute maximum and Absolute minimum values of function f given by f(x) = $\sin^2 x - \cos x$; $x \in [0,\pi]$

Answer:

We are given, f (x) = $\sin^2 x - \cos x$

Differentiating both sides w,r.t x

f (x) = $\sin^2 x - \cos x$

$\Rightarrow$ f '(x) = 2sinxcosx – sin x

Considering f'(x) = 0

f '(x) = 2sinxcosx – sin x

2sinxcosx – sin x = 0= sin x(2cos x +1) = 0

Either, sinx = 0, or

x = 0, π

(2cos x +1) = 0

$$\Rightarrow \cos x = -\frac{1}{2} = \cos\frac{2\pi}{3}$$

$$\Rightarrow x = \frac{2\pi}{3}$$

The values at x = 0, $\frac{2\pi}{3}, \pi$ are

f (x) = $\sin^2 x - \cos x$

f(0) = $\sin^2 0 - \cos 0 = 0 - 1 = -1$

$$f\left(\frac{2\pi}{3}\right) = \sin^2\frac{2\pi}{3} - \cos\frac{2\pi}{3} = \left(\frac{\sqrt{3}}{2}\right)^2 - \left(-\frac{1}{2}\right) = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$$

f(π) = sin2π - cosπ = 0 – (-1) = 0 + 1 = 1

Hence, the absolute maximum value is f(0) = -1 and absolute maximum is $f\left(\frac{2\pi}{3}\right) = \frac{5}{4}$

Question: 22

Show that the lines, $\vec{r} = \hat{i} + \hat{j} + \hat{k} + \lambda\left(\hat{i} - \hat{j} + \hat{k}\right)$, $\vec{r} = 4\hat{j} + 2\hat{k} + \mu\left(2\hat{i} - \hat{j} + 3\hat{k}\right)$ are coplanar. Also find the equation of plane containing theses lines.

Answer:

The given two lines are said to be coplanar, if they are parallel or intersect at some point. Rewriting these equations in Cartesian points, and finding the points as,

$$\frac{x-1}{1} = \frac{y-1}{-1} = \frac{z-1}{1} = \lambda$$

x = λ + 1

y = -λ + 1

z = λ + 1

$$\frac{x-0}{1} = \frac{y-4}{-1} = \frac{z-2}{1} = \mu \text{, or}$$

x = 2μ

y = -μ +4

z = 3μ+ 2

Let the lines intersect, then

λ + 1 = 2μ $\Rightarrow$ λ - 2μ = -1

-λ+1 = -μ +4 $\Rightarrow$ -λ + μ = 3

λ + 1 = 3μ + 2

λ - 3μ = 1

Solving, the first two equations

λ - 2μ = -1, or

-λ + μ = 3, or

μ= -2

Substituting μ= -2 in first equation,

λ - 2μ = -1

λ - 2(-2) = -1

λ = -5

Substitution the values of λ, and μ in t hird equation as,

λ - 3μ = 1, or

(-5) -3(-2) = 1, or

(-5) -3 (-2) = 1 – 5 + 6 = 1, or

1=1

To find the equation of the plane containing lines, we need to find intersection points and the normal to both the lines. On substituting the value of λ, The points are, x = 4, y = 6, z = -4

and the normal vector is calculated as,

$$b_1 x b_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & -1 & 3 \end{vmatrix} = \hat{i}(-3+1) - \hat{j}(3-2) + \hat{k}(-1+2) = -2\hat{i} - \hat{j} + \hat{k}$$

Therefore the equation of the plane is,
-2(x+4) -1(y-6) + 1 (z+4) = 0, or
-2x – 8-y+6 + z + 4 = 0, or
$2x + y - z + 2 = 0$

Question: 23

Let f :W →W defined as, $f(n) = \begin{Bmatrix} n-1 \text{ If n is odd} \\ n+1 \text{ If n is even} \end{Bmatrix}$. Show that f is invertible and also find the inverse of f.

Answer:

We have given, f: W → W

Defined as, $f(n) = \begin{Bmatrix} n-1 \text{ If n is odd} \\ n+1 \text{ If n is even} \end{Bmatrix}$

For one – one
f(n) = f(m)
n -1 = m + 1
n - m = 2

However, this is impossible.

Similarly, the possibility of n being even and m being odd can be ignored under a similar argument. Therefore, both n, and m must be either odd or even.

Now, if both n, and m are odd, then, we have
f(n) = f(m)
n -1 = m – 1, hence,
n =m

Again, if both m and n are even, then, we have
f(n) = f(m)
n + 1 = m + 1
n =m

Therefore, f is one-one.

It is clear that any odd number 2r + 1 in co-domain N is the image of 2r in domain N, and any even number 2r in co-domain N is the image of 2r + 1 in domain N. Therefore, f is onto. Hence, f is an invertible function.

Let us, define g: W →W

$$g(m) = \begin{Bmatrix} m-1 \text{ If n is odd} \\ m+1 \text{ If n is even} \end{Bmatrix}$$

Now, I when n is odd
gof(n) = g(f(n)) = g(n-1) = n – 1 + 1= n

And, when n is even,
gof(n) = g(f(n)) = g(n+1) = n+ 1 – 1 = n

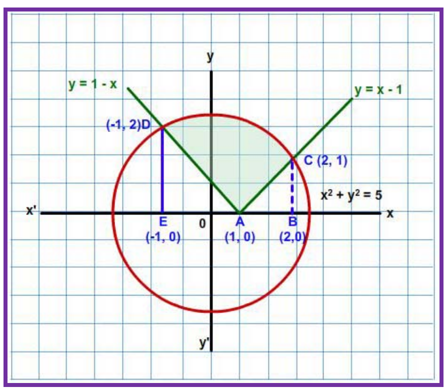
Similarly, when m is odd
fog(m) = f(g(m)) = g(m-1) = m – 1 + 1 =m

And, when m is even
fog(m) = f(g(m)) = g(m+1) = m + 1 -1 =m

Hence, gof = I_w, and fog = I_w. Thus, f is invertible and inverse of f is given by f^{-1} = g which is same as f. so, the inverse of f is f itself.

Question: 24

Sketch the region bounded by $y=\sqrt{5-x^2}$ and $y = |x-1|$, and find its area using integration $x^2+y^2=5$, and $y = \begin{Bmatrix} 1-x & x<1 \\ x-2 & x>1 \end{Bmatrix}$ figure is as follows:



Answer:

Point of intersection is C(2,1), and D(-1,2). Therefore,

$$\text{Area} = \int_{-1}^{2}\sqrt{5-x^2}\,dx - \int_{-1}^{1}(1-x)dx - \int_{-1}^{2}\sqrt{(x-1)}dx$$

$$= \left\{\left(\frac{x}{2}\sqrt{5-x^2}\right)+\left(\frac{5}{2}\sin^{-1}\frac{x}{\sqrt{5}}\right)\right\}_{1}^{2} - \left\{x-\frac{x^2}{2}-x\right\}_{2}^{2}$$

$$= \left\{1+\left(\frac{5}{2}\sin^{-1}\frac{2}{\sqrt{5}}\right)+\frac{1}{2}.2-\left(\frac{5}{2}\sin^{-1}\frac{-1}{\sqrt{5}}\right)\right\}-\left(1-\frac{1}{2}+1+\frac{1}{2}\right)-\left(2-2-\frac{1}{2}+1\right)$$

$$= \left[\left\{\frac{5}{2}\sin^{-1}\frac{2}{\sqrt{5}}\sin^{-1}\left(\frac{-1}{\sqrt{5}}\right)\right\}-\frac{1}{2}\right]$$

Question: 25

Two numbers are selected at random (without replacement) from first six positive integers. Let X denotes the larger of the two numbers obtained. Find the probability distribution of X, the mean and variance of this distribution.

Answer:

The greater of two numbers can be 1, 2,3,4,5, and 6, therefore, X can be 1,2,3,4,5,6.

$$P(1) = \frac{1}{36}$$

$$P(2) = \frac{3}{36}$$

$$P(3) = \frac{5}{36}$$

$$P(4) = \frac{7}{36}$$

$$P(5) = \frac{9}{36}$$

$$P(6) = \frac{11}{36}$$

The probability distribution of X is as follows:

X	1	2	3	4	5	6
Probability	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

Expectation (mean) is calculated as:

$$= \left\{1 \times \left(\frac{1}{36}\right)\right\} + \left\{2 \times \left(\frac{3}{36}\right)\right\} + \left\{3 \times \left(\frac{5}{36}\right)\right\} + \left\{4 \times \left(\frac{7}{36}\right)\right\} + \left\{5 \times \left(\frac{9}{36}\right)\right\} + \left\{6 \times \left(\frac{11}{36}\right)\right\}$$

$$= \frac{1+6+15+28+45+66}{36} = \frac{161}{36} = 4.47$$

Variance is:

$=[E(X^2)] - [E(X)]^2$

$$= \left\{1^2 \times \left(\frac{1}{36}\right)\right\} + \left\{2^2 \times \left(\frac{3}{36}\right)\right\} + \left\{3^2 \times \left(\frac{5}{36}\right)\right\} + \left\{4^2 \times \left(\frac{7}{36}\right)\right\} + \left\{5^2 \times \left(\frac{9}{36}\right)\right\} + \left\{6^2 \times \left(\frac{11}{36}\right)\right\}$$

$$= \left(\frac{1+12+45+112+225+396}{36}\right) - \left(\frac{161}{36}\right)^2 = \left(\frac{791}{36}\right) - \left(\frac{161}{36}\right)^2 = \frac{2555}{1296} = 1.97$$

Question: 26

Minimize, and maximize Z= 5x +2y subject to the following constraints, $x - 2y \le 2$, $3x + 2y \le 12$, and $-3x + 2y \le 3$ at $x \ge 0$, and $y \ge 0$

Answer:

Let us consider the following lines:

x - 2y = 2, 3x + 2y =12, and -3x + 2y = 3

Find the shaded region

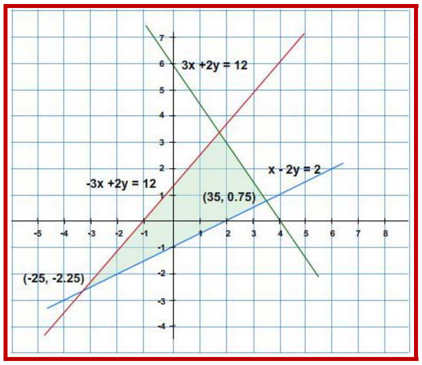
$-3x + 2y \le 3$, $0-2(0) \le 2$, and $0 \le 2$

This is true, so the region towards origin should be shaded for $x - 2y \le 2$. For $3x + 2y \le 12$, $3(0) + 2(0) \le 12$, and $0 \le 12$

This is true, so the region towards origin should be shaded for $3x + 2y \le 12$. For

$-3x + 2y \leq 3$, $-3(0) + 2(0) \leq 3$, and $0 \leq 3$

This is true, so the region towards origin should be shaded for $-3x + 2y \leq 3$. The graph of the following lines with shaded region is:



$Z = 5x + 2y$, $x - 2y = 2$, $3x + 2y =12$, and $-3x + 2y = 3$

Solving, $x – 2y = 2$, and $3x + 2y = 12$

We get,
$4x = 14$, or
$x = \frac{14}{4} = 3.5$, and
$x -2y = 2$
$x = 3.5 – 2y = 2$, and
$y = 0.75$

93,5, 0.75)

Solving, $3x + 2y = 12$, and $-3x + 2y = 3$

We get,
$3x + 2y = 12$, or
$-3x + 2y = 3$, or
$4y = 15$, or
$y = 3.75$

$-3x + 2 (3.75) = 3$, or
$-3x = -4.5$, or
$x = 1.5$ (1.5, 3.75)

Solving,
$x-2y = 3$, or
$-3x + 2y = 3$, or
$-2x = 5$, or
$x = -2.5$

$-2.5 – 2y = 2$, or
$-4.5 = 2y$, or
$y = -2.25$ (-2.5, -2.25)

Corner Points	Z value
(3.5, 0.75)	5x + 2y= 5(3.5) + 2(0.75) =17.5 + 1.5= 19
(1.5, 3.75)	5x + 2y = 5(1.5) + 2(0.75) = 7.5 + 7.5= 15

Therefore, Z is maximum at (3.5,0.75),and minimum at (1.5, 3.75). Hence, maximum, and minimum values are 19, and 15 respectively.