2009

Question

Section: A

Questions number 1 to 10 carry 1 mark each.

Question: 1 [1]

Evaluate, $\int \frac{1}{x + x\log x}$

Answer:

$$\text{Let, } I = \int \left(\frac{1}{x + x\log x} \right) dx = \int \frac{dx}{x(1+\log x)}$$

$$\text{Let, } 1 + \log x = t$$

$$\left(\frac{1}{x} \right) dx = dt$$

$$\therefore \ I = \int \left(\frac{dt}{t} \right) = \log|t| + C = \log|1 + \log x| + C$$

Question: 2 [1]

Evaluate: $\int_1^1 \left(\frac{1}{\sqrt{2x+3}} \right) \times dx$

Answer:

$$\int_0^1 \left(\frac{1}{\sqrt{2x+3}} \right) dx = \left\{ \int_0^1 (2x+3)^{-\frac{1}{2}} \right\} \times dx = \left. \frac{(2x+3)^{\frac{1}{2}}}{\left(\frac{1}{2} \times 2 \right)} \right]_0^1 = 5^{\frac{1}{2}} - 3^{\frac{1}{2}} = \sqrt{5} - \sqrt{3}$$

Question: 3 [1]

If the binary operation *, defined on Q, is defined as a * b = 2a + b - ab, for all a, b e Q, find the value of 3 * 4.

Answer:

Given binary operation is

a * b = 2a + b – ab

$\therefore$ 3 *4 = 2 x 3 + 4 – 3 x 4 $\Rightarrow$ 3 * 4 = -2

Question: 4 [1]

If $\begin{pmatrix} y+2x & 5 \\ -x & 3 \end{pmatrix} = \begin{pmatrix} 7 & 5 \\ -2 & 3 \end{pmatrix}$, find the value of y.

Answer:

Using equality of two matrices in, $\begin{pmatrix} y+2x & 5 \\ -x & 3 \end{pmatrix} = \begin{pmatrix} 7 & 5 \\ -2 & 3 \end{pmatrix}$

We get

y + 2x = 7

-x = -2 $\Rightarrow$ x = 2

$\therefore$ y + 2(2) = 7 $\Rightarrow$ y = 3

Question: 5 [1]

Find a unit vector in the direction of $\vec{a} = 2\hat{i} - 3\hat{j} + 6k$.

Answer:

Unit vector, $\vec{a} = \left(\dfrac{\vec{a}}{|\vec{a}|}\right)$

Then, $|\vec{a}| = \sqrt{(2)^2 + 3 + (6)^2} = \sqrt{4+3+36} = \sqrt{43} = 6.557 \Rightarrow \dfrac{\vec{a} = 2\hat{i} - 3\hat{j} + 6k}{6.557}$

Question: 6 [1]
Find the direction cosines of the line passing through the following points :(-2, 4, -5), (1, 2, 3)

Answer:

Step: 1

Let, $\overrightarrow{OA} = -2\hat{i} + 4\hat{j} - 5\hat{k}$

$\overrightarrow{OB} = \hat{i} + 2\hat{j} + 3\hat{k}$

$\therefore \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \left(\hat{i} + 2\hat{j} + 3\hat{k}\right) - \left(-2\hat{i} + 4\hat{j} - 5\hat{k}\right) = 3\hat{i} - 2\hat{j} + 8k$

Step: 2

$\overrightarrow{AB} = \sqrt{(3)^2 + (-2)^2 + (8)^2} = \sqrt{9+4+64} = \sqrt{77}$

Step 3

The direction cosines are, $\left(\dfrac{3}{\sqrt{77}}\right), \left(\dfrac{-2}{\sqrt{77}}\right), \left(\dfrac{8}{\sqrt{77}}\right)$

Question: 7 [1]

If $A = \left(a_{ij}\right) = \begin{pmatrix} 2 & 3 & -5 \\ 1 & 4 & 9 \\ 0 & 7 & -2 \end{pmatrix}$, and $B = \left(a_{ij}\right) = \begin{pmatrix} 2 & 1 & -1 \\ -3 & 4 & 4 \\ 1 & 5 & 2 \end{pmatrix}$ then find $a_{22} + b_{21}$.

Answer:
$a_{22} = 4$, $b_{21} = -3$
$a_{22} + b_{21} = 4 - 3 = 1$

Question: 8 [1]
If $|\vec{a}| = \sqrt{3}, |\vec{b}| = 2$, and $\vec{a}.\vec{b} = \sqrt{3}$, find the angle between $\vec{a}$ and $\vec{b}$.

Answer:

As we know, $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\cos\theta \Rightarrow \sqrt{3} = \sqrt{3}(2)\cos\theta \Rightarrow \dfrac{1}{2} = \cos\theta \Rightarrow \theta = \dfrac{\pi}{3}$

Question: 9 [1]

If $A = \begin{pmatrix} 1 & 2 \\ 4 & 2 \end{pmatrix}$, then find the value of k if $|2A| = k|A|$.

Answer:

Given, $A = \begin{pmatrix} 1 & 2 \\ 4 & 2 \end{pmatrix} \Rightarrow 2A = \begin{pmatrix} 2 & 4 \\ 8 & 4 \end{pmatrix}$

$\therefore |2A| = 8 - 32 = -24 \Rightarrow |A| = 2 - 8 = -6 \Rightarrow -24 = k(-6) \Rightarrow 4 = k$

Question: 10 [1]

Write the principal value of $\tan^{-1}\left\{\tan\left(\frac{3\pi}{3}\right)\right\}$

Answer:

$\tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \tan^{-1}\left\{\tan\left(\pi - \frac{\pi}{4}\right)\right\} = \tan^{-1}(-1) = -\left(\frac{\pi}{4}\right)$

$\therefore \text{Principal value of, } \tan^{-1}\left\{\tan\left(\frac{3\pi}{4}\right)\right\} = \frac{-\pi}{4}.$

Section: B

Question: 11 [4]

Evaluate, $\int \frac{(\cos x)dx}{(2+\sin x)(3+4\sin x)}$

Answer:

$\text{Let}, I = \int \frac{(\cos x)dx}{(2+\sin x)(3+4\sin x)}$ where, $\sin x = t$, and cos x dx = dt. Then

$\therefore I = \int \frac{dt}{(2+t)(3+4t)}$

$\text{Let}, \frac{1}{\{(2+t)(3+4t)\}} = \left\{\left(\frac{A}{2+t}\right)+\left(\frac{B}{3+4t}\right)\right\} \Rightarrow 1 = A(3+4t)+B(2+t) \Rightarrow 3A+2B+1$

$4A+B=0 \Rightarrow B=-4A$

$\therefore 3A-8A=1$

Hence,

$$A = -\frac{1}{5} \Rightarrow B = \frac{4}{5} \Rightarrow I = \int\left\{\frac{dt}{(2+t)(3+4t)}\right\} = \left\{\frac{-1}{5}\times\int\left(\frac{dt}{2+t}\right)\right\}+\left\{\frac{4}{5}\times\int\left(\frac{dt}{3+4t}\right)\right\}$$

$$= \frac{-1}{5}(\log|2+t|)+\frac{4}{5}\left(\frac{\log|3+4t|}{4}\right)+C = \frac{-1}{5}(\log|2+\sin x|)+\frac{1}{5}(\log|3+4\sin x|)+C$$

$$= \frac{-1}{5}\times\left(\log\left|\frac{3+4\sin x}{2+\sin x}\right|\right)+C$$

OR

Evaluate, $\int x^2 \times \cos^{-1} x dx$

Answer:

$$= \left\{(\cos^{-1}x)\frac{x^3}{3}\right\} - \left\{\int\left(\frac{x^{dx}}{\sqrt{1-x^2}}\right)\times\left(\frac{x^3}{3}\right)dx\right\} \qquad \text{(Integrating by sections)}$$

$$= \left\{\frac{x^3}{3}(\cos^{-1}x)\right\} + \left\{\frac{1}{3}\times\int\left(\frac{x^3 dx}{\sqrt{1-x^2}}\right)\right\} = \left\{\frac{x^3}{3}(\cos^{-1}x)\right\}+\left\{\frac{1}{3}(I_1)\right\}$$

In I_1, let $1 - x^2 = t$ so that $-2x\,dx = dt$
In I_1, let $1 - x^2 = t$

$$\therefore I_1 = -\frac{1}{2}\times\int\left(\frac{1-t}{\sqrt{t}}\right)dt = -\frac{1}{2}\times\int\left(\frac{1}{\sqrt{t}}-\sqrt{t}\right)dt = -\frac{1}{2}\times\left\{2\sqrt{t}-\left(\frac{2}{3}\right)t^{\frac{3}{2}}\right\}+C'$$

$$\therefore I = \left\{\frac{x^3}{3}(\cos^{-1}x)\right\}-\left\{\frac{1}{3}(1-x^2)^{\frac{3}{2}}\right\}+C'$$

Question: 12 [4]

Show that the relation R in the set of real numbers, defined as R = {(a, b) : a ≤; b^2} is neither reflexive, nor symmetric, nor transitive.

Answer:

It can be observed that R is not reflexive. Now, (1, 2) ∈ R as $1 < 2^2$. But, 2 is not less than 1^2.

∴ (2, 1) ∉ R

∴ R is not symmetric.

Now, (5, 3), (3, 2) ∈ R (as $5 < 3^2 = 9$, and $3 < (2)^2 = 4$)

But, $5 > (2)^2 = 4$

∴ (5, 2) ∉ R

∴ R is not transitive. Therefore, R is neither reflexive, nor symmetric, nor transitive.

Question: 13 (**)

If $\log(x^2+y^2) = 2\tan^{-1}\left(\frac{y}{x}\right)$, then show that, $\frac{dy}{dx} = \frac{x+y}{x-y}$

Answer:

OR

If x = a (cos t + ta sin t), and y = a (sin t – t cos t), then find $\left(\frac{d^2y}{dx^2}\right)$.

Answer:

Given is x = a (cos t + ta sin t), and y = a(sin t – t cos t)

$$\Rightarrow \frac{dy}{dx} = a\{-\sin t + (t.\cos t + \sin t.1)\} = at.\cos t ....(i)$$

$$\Rightarrow \frac{dy}{dx} = a\left[\cos t - \{t.(-\sin t) + \cos.1\}\right] = at.\sin t...........(ii)$$

from (i), and (ii), we get, $\frac{dy}{dx} = \left\{\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}\right\} = \frac{at.\sin t}{at.\cos t} = \tan t$

Now, $\left(\frac{d^2y}{dx^2}\right) = \sec^2 t\left(\frac{dt}{dx}\right) \Rightarrow \frac{d^2y}{dx^2} = \sec^2 t\left(\frac{1}{at.\cos t}\right) \Rightarrow \frac{d^2y}{dx^2} = \frac{\sec^3 t}{at}$

Question: 14 [4]

Find the equation of the tangent to the curve $y = \sqrt{4x-2}$ which is parallel to the line 4x – 2y + 5 = 0.

Answer:

Since the tangent to the given curve is parallel to the line 4x - 2y + 5 = 0

Slope of the tangent = Slope of the given line = $\left(\frac{-4}{-2}\right) = 2$

$\left(\frac{dy}{dx}\right)$, the slope of the tangent, $\frac{d\sqrt{3x-2}}{dx} = \frac{1\times 3}{2\sqrt{3x-2}} = \frac{3}{2\sqrt{3x-2}} = \frac{3}{2\sqrt{3x-2}} = 2 = \frac{3}{4}\sqrt{3x-2}$

$= 3x-2 = \frac{9}{16}$ (squaring both sides)

48x - 32 = 9

$x = \frac{41}{48}$

$y = \sqrt{3x-2} = \sqrt{\left\{3\times\left(\frac{41}{48}\right)\right\}-2} = \pm\left(\frac{3}{4}\right)$, then

$(x,y) = \left(\frac{41}{3}\right)\times\left(\frac{3}{4}\right)$

Equation of the tangent to the curve,

$y = \sqrt{3x-2}$ at $\left(\frac{41}{3}\right)\left(\frac{3}{4}\right)$

or, $y-\frac{3}{4} = 2\left(x-\frac{41}{3}\right)$.

Simplified equation is, 48 x - 24y = 23.

Question: 15 [4]

Prove the following: $\tan^{-1}\left(\frac{1}{4}\right)+\tan^{-1}\left(\frac{2}{9}\right) = \frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)$

Answer:

L.H.S.= We shall rewrite the question as prove, $2\times\left\{\tan^{-1}\left(\frac{1}{4}\right)+\tan^{-1}\left(\frac{2}{9}\right) = \cos^{-1}\left(\frac{3}{5}\right)\right\}$

By taking, $x = \left(\frac{1}{4}\right)$, and $y = \left(\frac{2}{9}\right)$ we get, $\frac{x+y}{1-xy} = \left[\frac{\left(\frac{1}{4}+\frac{2}{9}\right)}{\left\{1-\left(\frac{1}{4}\times\frac{2}{9}\right)\right\}}\right] = \left\{\left(\frac{17}{36}\right)\times\left(\frac{36}{34}\right)\right\} = \frac{17}{34} = \frac{1}{2}$

Substituting in the above formula we get, L.H.S. $= \left\{\tan^{-1}\left(\frac{1}{4}\right)+\tan^{-1}\left(\frac{2}{9}\right)\right\} = \tan^{-1}\left(\frac{1}{2}\right)$

By taking, $x = \left(\frac{1}{2}\right), \left(\frac{2x}{1-x^2}\right) = \frac{\left(2\times\frac{1}{2}\right)}{\left(1-\frac{1}{4}\right)} = \frac{4}{3}$

Substituting in the above formula of $2\tan^{-1}x$ we get, $2\tan^{-1}\left(\frac{1}{2}\right) = \tan^{-1}\left(\frac{4}{3}\right)$

$x = \left(\frac{4}{3}\right)$ we get, $\frac{1}{\left(\sqrt{1+x^2}\right)} = \frac{1}{\left(\sqrt{1+\frac{16}{9}}\right)} = \frac{3}{5}$

By substituting in the formula, $\tan^{-1}x = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$ we get, $\tan^{-1}\left(\frac{4}{3}\right) = \cos^{-1}\left(\frac{3}{5}\right)$

Then L.H.S. becomes, $2\tan^{-1}\left(\frac{1}{2}\right)=\tan^{-1}\left(\frac{4}{3}\right)=\cos^{-1}\left(\frac{3}{5}\right)$=R.H.S.

Question: 16 [4]

Find the angle between the line $\frac{x+1}{2}=\frac{3y+5}{9}=\frac{3-z}{-6}$,and the plane 10x + 2y -11z = 3.

Answer:

Given line can be rearranged to get, $\frac{x-(-1)}{2}=\frac{\left\{y-\left(-\frac{5}{3}\right)\right\}}{3}=\frac{z-3}{6}$

Its direction ratios are 2, 3, 6. Direction ratios of normal to the plane 10x + 2y -11z = 3 are 10, 2, -11. Angle between the line, and plane

$$\sin\theta=\frac{(2\times10)+(3\times2)+\{6(-11)\}}{\left(\sqrt{4+9+36}\right)\times\left(\sqrt{100+4+121}\right)}=\left\{\frac{(20+6-66)}{(7\times15)}\right\}=\left(\frac{-40}{105}\right)\sin\theta=-\left(\frac{8}{21}\right)\text{or } \theta=\sin^{-1}-\left(\frac{8}{21}\right)$$

Question: 17 [4]

Solve the differential equation: (x^3+y3)dy – x2y dx = 0.

Answer:

$\left(x^3+y\right)^3 dy-x^{2ydx=0}$ is rearranged as $\frac{dy}{dx}=\frac{x^2y}{x^3+y^3}$

It is a homogeneous differential equation.

Let, $\left(\frac{y}{x}\right)=v\Rightarrow y=vx\left(\frac{dy}{dx}\right)=v+x\left(\frac{dv}{dx}\right)$

$$\therefore v+x\left(\frac{dv}{dx}\right)=\left(\frac{-v^4}{1+v^3}\right)\Rightarrow x\left(\frac{dv}{dx}\right)=\left\{\left(\frac{v}{1+v^3}\right)-v\right\}=\frac{-\left(v^4\right)}{\left(1+v^3\right)}\Rightarrow\left(\frac{1+v^3}{v^4}\right)dv=-\left(\frac{dx}{x}\right)$$

Integrating both sides, we get

$$\int\left(\frac{1}{v^4}+\frac{1}{v}\right)dv=-\int\left(\frac{dx}{x}\right)\Rightarrow-\frac{1}{3v^3}\times\left(\log|v|\right)=-\left(\log|x|\right)+C\Rightarrow-\left(\frac{x^3}{3y^3}\right)\times\left(\log\left|\frac{y}{x}\right|\right)=-\left(\log|x|\right)+C$$

$\Rightarrow-\left(\frac{x^3}{3y^3}\right)\times\left(\log|y|\right)=C$ is the solution of the given differential equation.

Question: 18 [4]

Find the particular solution of the differential equation $\frac{dy}{dx}$ + y cot x = 4x cosec x, (x ≠ 0), given that y = 0, when, x = $\frac{\pi}{2}$

Answer:

<u>Step</u>: 1

Given, $\frac{dy}{dx}+y\cot x=4x\operatorname{cosec} x$

This is a first order linear differential equation. Here p=cot x, and Q=4xcosec x

$\int pdx = \int cotx = log\ sinx|$

Hence, $I.F = e^{\log|\sin x|} = \sin x$

Step: 2

Hence the solution is $y \sin x = \int \{(4x \text{ cosec } x)\times(\sin x)dx\} + C$

or, $\text{cosec}\, x = \frac{1}{\sin x}$, then

$$y \sin x = \int 4xdx + C = \frac{4x^2}{2} + C = 2x^2 + C.........(1)$$

Step 3

To evaluate the value of C, let us substitute, $x = \frac{\pi}{2}$, and $y = 0$

$$\left\{0\times\sin\left(\frac{\pi}{2}\right)\right\} = \left\{2\times\left(\frac{\pi}{2}\right)^2\right\} + C$$

$$0 = \frac{\pi^2}{2} + C$$

$$C = -\left(\frac{\pi^2}{2}\right)$$

Substituting the value of C in equation (1) we get, $y \sin x = 2x^2 - \frac{\pi^2}{2}$. This is the required solution.

Question: 19 [4]

Using properties of determinants and prove, $\begin{vmatrix} (a^2+1) & ab & ac \\ ab & (b^2+1) & bc \\ ca & cb & (c^2+1) \end{vmatrix} = 1 + a^2 + b^2 + c^2$

Answer:

Let, $|A| = \begin{vmatrix} (a^2+1) & ab & ac \\ ab & (b^2+1) & bc \\ ca & cb & (c^2+1) \end{vmatrix}$, Apply, $C_1 \to aC_1$, $C_2 \to bC_2$, $C_3 \to cC_3$

Let, $|A| = \left(\frac{1}{abc}\right) \times \begin{vmatrix} a(a^2+1) & ab^2 & c^2a \\ a^2b & b(b^2+1) & c^2b \\ a^2c & b^2c & c(c^2+1) \end{vmatrix}$,

Taking a, b, c common respectively from R_1, R_2, and R_3 we get,

$$|A| = \left(\frac{abc}{abc}\right) \times \begin{vmatrix} (a^2+1) & b^2 & c^2 \\ a^2 & (b^2+1) & c^2 \\ a^2 & b^2 & (c^2+1) \end{vmatrix} \text{ by applying, } C_1 \to C_1 + C_2 + C_3$$

Question: 20 [4]

The probability that A hits a target is $\frac{1}{3}$,and the probability that B3 hits it is $\frac{2}{5}$. If each one of A, and B shoots at the target, what is the probability that (**)

i. the target is hit?

Answer:

Let P(A) = Probability that A hits the target = $\frac{1}{3}$

P(B) = Probability that B hits the target = $\frac{2}{5}$

(i) P(target is hit) = P(at least one of A, B hits)
= 1 - P (none hits)

$$1-\frac{2}{3}\times\frac{3}{5}=\frac{9}{15}=\frac{3}{5}$$

ii. exactly one of them hits the target?

Answer:

P(exactly one of them hits) = $P\left(A \,\&\, \overline{B} \text{ or } \overline{A} \,\&\, B\right)$

$$= P(A) \times P\left(\overline{B}\right) + P(\overline{A}).P(B)$$

$$\frac{1}{3}\times\frac{3}{5}+\frac{2}{3}\times\frac{2}{5}=\frac{7}{15}$$

Question: 21 [4]

Find $\frac{dy}{dx}$, if $y^x + x^y = a^b$, where a, b are constants.

Answer:

$y^x + x^y = a^b$ …………………(i)
let $v = y^x$
$u = x^y$
Taking log on either side of the two equation, we get
$\log v = x \log y$, $\log u = y \log x$
Differentiating w.r.t.x, we get

$$\frac{1}{v}\frac{dv}{dx} = x.\frac{1}{y}\frac{dy}{dx} + \log y,\ \frac{1}{u}\frac{du}{dx} = \frac{y}{x} + \log x.\frac{dy}{dx}$$

$$\Rightarrow \frac{dv}{dx} = y^x\left[\frac{x}{y}\frac{dy}{dx} + \log y\right], \frac{du}{dx} = x^y\left[\frac{y}{x} + \log x.\frac{dy}{dx}\right]$$

From (i), we have

$$u + v = a^b$$

$$\Rightarrow \frac{du}{dx} + \frac{dv}{dx} = 0$$

$$\Rightarrow y^x\left[\frac{x}{y}\frac{dy}{dx}+\log y\right]+x^y\left[\frac{y}{x}+\log x.\frac{dy}{dx}\right]=0$$

$$\Rightarrow y^x.\frac{x}{y}\frac{dy}{dx}+x^y.\log x\frac{dy}{dx}=-y^x\log y-x^y.\frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx}=\frac{-y^x\log y-x^{y-1}y}{y^{x-1}x+x^y.\log x}$$

Question: 22 [4]

If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $\vec{a}.\vec{b}=\vec{a}.\vec{c}$ and $\vec{a}\times\vec{b}=\vec{a}\times\vec{c}, \vec{a}\neq 0$, then show that $\vec{b}=\vec{c}$

Answer:

Given $\vec{a}.\vec{b}=\vec{a}.\vec{c}$

$\Rightarrow \vec{a}.\vec{b}-\vec{a}.\vec{c}=0 \;\Rightarrow \vec{a}.\left(\vec{b}-\vec{c}\right)=0 \Rightarrow$ either $\vec{b}=\vec{c}$ or $\vec{a}\perp\vec{b}-\vec{c}$

Also given $\vec{a}\times\vec{b}=\vec{a}\times\vec{c}$

$\Rightarrow \vec{a}\times\vec{b}=\vec{a}\times\vec{c}=0 \;\Rightarrow \vec{a}\times\left(\vec{b}-\vec{c}\right)=0$

$\Rightarrow \vec{a}||\vec{b}-\vec{c}$ or $\vec{b}=\vec{c}$

Section: C

Questions number 23 to 29 carry six marks each.

Question: 23 [6]

One kind of cake requires 200 g of flour, and 25 g of fat, and another kind of cake requires 100 g of flour, and 50 g of fat. Find the maximum number of cakes which can be made from 5 kg of flour, and 1 kg of fat assuming that there is no shortage of the other ingredients used in making the cakes. Formulate the above as a linear programming problem, and solve graphically.

Answer:

We are given 5kg of flour, and 1 kg of fat. Let x be the number of Cakes of Type-1, and y be the number of cakes of Type-2 that we can make. Our problem is to maximize x + y. Clearly, x, y ≥ 0. Let us construct the following table from the given data (all numbers converted from kg to g)

	Cake1	Cake2	Requirements
Flour A	200	100	5000
Flour B	25	50	1000

$(1): 200x + 100y \leq 5000 \rightarrow 2x + y \leq 50$

$(2): 25x + 50y \leq 1000 \rightarrow x + 2y \leq 40$

$(3): x \leq 0, (4): y \leq 0$

Plot the straight lines, 2x+y = 50, and x+2y = 40.

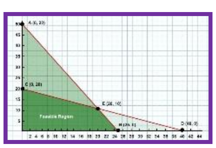
First draw the graph of the line 2x+y = 50. If x = 0, y = 50, and if y = 0, x = 25. So, this is a straight line between (0,50), and (25,0). At (0,0), in the inequality, we have 0 + 0 = 0 which is≤ 0. So the area associated with this inequality is bounded towards the origin.

Similarly, draw the graph of the line x + 2y = 40. If x = 0, y = 20, and if y =0, x = 40. So, this is a straight line between (0,20), and (40,0). At (0,0), in the inequality, we have 0 + 0 = 0 which is ≤ 0. So the area associated with this inequality is bounded towards the origin. Finding the feasible region: We can see that the feasible region is bounded, and in the first quadrant.

On solving the equations, 2x + y = 50, and x + 2y = 40, we get,
2 (40 – 2y) + y = 50 →→ 80 – 4y + y = 50 →→ -2y = -30 →→ y = 10.

If y = 10, then x = (50 – 10)/2 = 20 ⇒x=20, y=10⇒x=20, y=10

Therefore, the feasible region has the corner points (0,0), (0,20), (20, 10), (25,0) as shown in the figure.



Solving the objective function using the corner point method, the values of Z at the corner points are calculated as follows:

Corner Point	x+y
O (0,0)	0
C(0,2)	20
E(20,10)	30 (Max Value)
B(25,0)	25

The maximum number of cakes we can make is 30, 20 of one kinds, and 10 of another.

Question: 24 [6]

Using integration, find the area of the region, $\{(x, y) : 9x^2 + \cdot y^2 \geq 36, \text{ and } 3x + y \geq 6\}$

Answer:

Given region is $\{(x, y): 9x^2 + y^2 \leq 36, \text{ and } 3x + y \geq 6\}$

We draw the curves corresponding to equations,

$9x^2 + y^2 = 36$, or

$\frac{x^2}{4} + \frac{y^2}{9} = 1$, and

$3x + y = 6$

The curves intersect at (2, 0), and (0, 6)

$\therefore$ Shaded area is the area enclosed by the two curves, and is,

$$= \int_0^2 \sqrt{9\left(1-\frac{x^2}{4}\right)}\,dx - \int_0^2 (6-3x)\,dx \; = 3\left[\left(\frac{x}{4}\times\sqrt{4-x^2}\right)+\left(\frac{4}{2}\times \sin^{-1}\frac{x}{2}\right)-\left(2x+\frac{x^2}{2}\right)\right]_0^2$$

$$= 3\left[\left(\frac{2}{4}\times\sqrt{4-4}\right)+\left(\frac{4}{2}\sin^{-1}\frac{2}{2}\right)-\left(4+\frac{4}{2}\right)-0\right] = 3\left\{2\left(\frac{\pi}{2}\right)-2\right\} = 3(\pi - 2) \text{ square units.}$$

Question: 25 [6]

Show that the lines are $\frac{x+3}{-3} = \frac{y-1}{1} = \frac{z-5}{5}$; $\frac{x+1}{-1} = \frac{y-2}{2} = \frac{z-5}{5}$ coplanar. Also find the equation of the plane containing the lines.

Answer:

Given lines are, $\frac{x+3}{-3} = \frac{y-1}{1} = \frac{z-5}{5}$, and $\frac{x+1}{-1} = \frac{y-2}{2} = \frac{z-5}{5}$

These lines will be coplanar if, $\begin{vmatrix} (x_2-x_1) & (y^2-y_1) & (z_2-z)_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$

$\therefore \begin{vmatrix} (-1+3) & (2-1) & (5-5) \\ -3 & 1 & 5 \\ -1 & 2 & 5 \end{vmatrix}$ = 2(5 - 10) - 1(-15 + 5) = 0. Hence lines are co-planar.

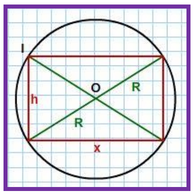
The equation of the plane containing two lines is, $\begin{vmatrix} (x+3) & (y-1) & (z-5) \\ -3 & 1 & 5 \\ -1 & 2 & 5 \end{vmatrix}$

$\Rightarrow$ -5(x + 3) + 10(y - 1) - 5(z - 5) = 0
$\Rightarrow$ -5x - 15 + 10y - 10 - 5z + 25 = 0
$\Rightarrow$ -5x + 10y - 5z + 0 = 0
$\Rightarrow$ -x + 2y - z = 0 or x - 2y + z = 0

Question: 26 [6]

Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume.

Answer:



Step: 1

Radius of the sphere = R = R
Let, hh be the diameter of the base of the inscribed cylinder.

Then, $h^2+x^2=(2R)^2$

$h^2+x^2=4R^2$ ....(1)

We know volume of a cylinder is $\pi r^2 h$, then, $V=\pi\left(\frac{x^2}{2}\right)^2\times h=\pi\left(\frac{x^4}{4}\right)\times h=\frac{1}{4}\times\left(\pi x^2\right)\times h$

Substituting the value of x^2 we get, $V=\left\{\frac{1}{4}\times\pi h\times\left(4r^2-h^2\right)\right\}$

From (1) we get, $x^2=4R^2-h^2$, then, $V=\left(\pi R^2 h\right)-\left(\frac{1}{4}\times\pi h^3\right)$

Step: 2

Differentiating with respect to x

$V=\left(\pi R^2 h\right)-\left(\frac{1}{4}\times\pi h^3\right)$

$\frac{dV}{dh}=\pi R^2-34\pi h^3=\pi\left\{R^2-\frac{3}{4}\left(h^2\right)\right\}=0\Rightarrow R^2=\frac{3}{4}\left(h^2\right)\Rightarrow h=\frac{2R}{\sqrt{3}}$

Step: 3

Also, $\frac{d^2V}{dh^2} = -\left\{\left(\frac{3}{4}\right)\times 2\pi h\right\} \Rightarrow -\left\{\frac{3}{2}\times(\pi h)\right\}$

At, $h = \frac{2R}{3} \Rightarrow \frac{d^2V}{dh^2} = \left(\frac{-3}{2}\right)\times\left\{\pi\left(\frac{2R}{\sqrt{3}}\right)\right\} =$ -ve $\Rightarrow$V is maximum at, $h = \frac{2R}{\sqrt{3}}$

Maximum volume at, $h = \frac{2R}{\sqrt{3}} \Rightarrow \frac{1}{4}\times\pi\times\left\{\left(\frac{2R}{\sqrt{3}}\right)\left(4R^2 - \frac{4R^2}{3}\right)\right\} \Rightarrow \frac{\pi R}{2\sqrt{3}}\times\left(\frac{8R^2}{3}\right) \Rightarrow \frac{4\pi R^3}{3\sqrt{3}}$ sq.units.

OR

Show that the total surface area of a closed cuboid with square base, and given volume, is minimum, when it is a cube.

Answer:

Let us assume that the square base has side x, and height h.

Let us assume that given volume of the cuboid, $\Rightarrow V = x^2h \Rightarrow h = \frac{V}{x^2}$

Now surface area of the cuboid, $SA(x) = 2x^2 + 4xh = 2x^2 + 4x\left(\frac{V}{x^2}\right) = 2x^2 + \frac{4V}{x}$

Taking first derivative to find critical points, we get, $Sa'(x) = 4x - \frac{4V}{x} = 0$

$\Rightarrow 4x = \frac{4V}{x^2} \Rightarrow x^3 = V \Rightarrow x = \sqrt[3]{V}$

Taking second derivative, SA'(x) = $4 + \frac{8V}{x^3} \Rightarrow SA''(\sqrt[3]{V}) = 4 + \frac{8V}{V} = 4 + 8 = 12 = +Ve$

Hence, x = $\sqrt[3]{V}$. Therefore, the surface area is minimum when h = x which implies that the cuboid is a cube.

Question: 27 [6]

Using matrices, solve the following system of linear equations:

3x- 2y + 3z = 8

2x + y- z = 1

4x- 3y + 2z = 4

Answer:

$A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}$, $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and $B = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$, Now, $|A|$ = 3(2-3) + 2(4+4) + 3(-6-4) = -17 $\neq$ 0

$\therefore$ A is non-singular, its inverse exists, now finding, $A^{-1} = \left(\frac{adjA}{|A|}\right)$

For adj(A), we need to find cofactors of each element of A, given below

C_{11}= -1 C_{12}= -8 C_{13}= -10

C_{21}= -5 C_{32}=9 C_{33}=7

$\therefore$ adj(A) = transpose of matrix obtained by replacing with its cofactors.

$$\therefore\ \operatorname{adj}(A)=\begin{bmatrix}-1 & -5 & -1\\ -8 & -6 & 9\\ -10 & 1 & 7\end{bmatrix}$$

$$\text{Now, } A^{-1}=\frac{\operatorname{adj}(A)}{|A|}=-\left(\frac{1}{17}\right)\begin{bmatrix}-1 & -5 & -1\\ -8 & -6 & 9\\ -10 & 1 & 7\end{bmatrix}$$

To solve for x, y, z, we have, X= $A^{-1}B$

$$X=-\left(\frac{1}{17}\right)\begin{bmatrix}-1 & -5 & -1\\ -8 & -6 & 9\\ -10 & 1 & 7\end{bmatrix}\begin{bmatrix}8\\ 1\\ 4\end{bmatrix}=-\left(\frac{1}{17}\right)\begin{bmatrix}-8-5-4\\ -64-6+36\\ -80+1+28\end{bmatrix}=-\left(\frac{1}{17}\right)\begin{bmatrix}-17\\ -34\\ -51\end{bmatrix}=\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}$$

$\therefore$ X = 1, Y = 2, and Z = 3

Question: 28 [6]

Evaluate, $\int\left\{\frac{x^4dx}{(x-1)\times(x^2+1)}\right\}$

Answer:

$$\text{Let, } I=\frac{x^2}{2}+x+\left(\frac{1}{2}\log|x-1|\right)-\left(\frac{1}{4}\log\left|x^2+1\right|\frac{1}{2}\tan\right)-1x+C$$

$$\text{Let, } I=\int\left\{\frac{x^4}{(x-1)\times(x^2-1)}\right\}dx$$

$$\text{Suppose, } \frac{x^4}{\{(x-1)\times(x^2+1)\}}=\frac{x^4-1+1}{\{(x-1)\times(x^2+1)\}}=x+1+\left\{\frac{1}{(x-1)\times(x^2+1)}\right\}$$

$$\text{Also let, } \frac{1}{(x-1)\times(x^2+1)}=\left(\frac{A}{x-1}\right)+\left(\frac{Bx+C}{x^2+1}\right)\Rightarrow 1 = A(x^2+1)+(Bx+C)(x-1)$$

Equating coefficients of similar terms.

A + B = 0
-B + C = 0 $\Rightarrow$ B = C
A - C = 1

$$\therefore A - B = 1\Rightarrow\left(\frac{A+B=0}{2A}\right)\Rightarrow A=\frac{1}{2}\Rightarrow\left(B-\frac{1}{2}\right)=C$$

$$\therefore I=\int\left[\left\{(x+1)+\frac{\left(\frac{1}{2}\right)}{(x-1)}\right\}-\left\{\left(\frac{1}{2}\right)\times\left(\frac{x+1}{x^2+1}\right)\right\}\right]dx$$

$$=\frac{x^2}{2}+x+\left(\frac{1}{2}\log|x-1|\right)-\left\{\frac{1}{4}\times\int\left(\frac{2x}{x^2+1}\right)dx\right\}-\left\{\frac{1}{2}\times\int\left(\frac{dx}{x^2+1}\right)\right\}-dx$$

OR

Evaluate, $\int_1^4 \left[|x-1|+|x-2|+|x-4|\right]dx$

Answer:

From given equation,

$$=\int_1^4 (x-1)dx+\int_1^4 -(x-2)dx+\int_1^4 (x-2)dx+\int_1^4 (x-4)dx$$

$$=\left.\frac{x^2}{2}-x\right]_1^4+\left[-\frac{x^2}{2}+2x\right]_1^2+\left.\frac{x^2}{2}-2x\right]_2^4+\left.\left(-\frac{x^2}{2}+4x\right)\right]_1^4$$

$$=\left(\frac{16}{2}-4-\frac{1}{2}+1\right)+\left(-2+4+\frac{1}{2}-2\right)+\left(\frac{16}{2}-8-2+4\right)+\left(-\frac{16}{2}+16+\frac{1}{2}-4\right)$$

$$=\left(5-\frac{1}{2}\right)+\frac{1}{2}+2+4+\frac{1}{2}=11+\frac{1}{2}=\frac{23}{2}$$

Question: 29 [6]

Two cards are drawn simultaneously (or successively without replacement) from a well shuffled pack of 52 cards. Find the mean, and variance of the number of red cards.

Answer:

Let, X be the number of red cards drawn. In the pack of 52 cards there are 26 red cards, and 26 black cards. We can draw 3 cards out of 52 in ${}^{52}C_3$ ways;

Scenario 1: P (X = 0) = P (no red cards drawn) $=\frac{\left({}^{26}C_0\times{}^{26}C_3\right)}{{}^{52}C_3}=\frac{2}{17}$

Scenario 2: P (X = 1) = P (one red card drawn) $=\frac{\left({}^{26}C_1\times{}^{26}C_2\right)}{{}^{52}C_3}=\frac{13}{34}$

Scenario 3: P (X = 2) = P (one red card drawn) $=\frac{\left({}^{26}C_2\times{}^{26}C_1\right)}{{}^{52}C_3}=\frac{13}{34}$

Scenario 4: P (X = 2) = P (one red card drawn) $=\left\{\frac{\left({}^{26}C_3\times{}^{26}C_0\right)}{{}^{52}C_3}\right\}=\frac{2}{17}$

∴ Probability distribution of number of red cards is given by,

x_i	0	1	2	3	
$P(x_i)$	$\frac{2}{17}$	$\frac{13}{34}$	$\frac{13}{34}$	$\frac{2}{17}$	
$x.P(x_i)$	0	$\frac{13}{34}$	$\frac{26}{34}$	$\frac{2}{17}$	$\sum x_i \times P(x_i)=\frac{3}{2}$

⇒ Mean of the distribution is, $\sum x_i.P(x_i)=\frac{3}{2}$