## 2014

**Question**

## Section: A

**Question: 1** [1]

Let R = {(a, $a^3$ ) : a is a prime number less than 5} be a relation. Find the range of R.

**Answer:**

Here R = {(a, $a^3$ ) : a is a prime number less than 5} $\Rightarrow$ R = {(2, 8), (3, 27)}. Hence range of R = {8, 27}.

**Question: 2** [1]

Use elementary column operations $C_2 \rightarrow C_2 - 2C_1$ in the matrix equation $\begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

**Answer:**

Column operation is to be done on $2^{nd}$ matrix.

Given $\begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

$C_2 \rightarrow C_2 - 2C_1$

$\Rightarrow \begin{bmatrix} 4 & -6 \\ 3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & -4 \\ 1 & -1 \end{bmatrix}$

**Question: 3** [1]

Find the angle between the lines $\vec{r} = 2\hat{i} - 5\hat{j} + k, \lambda(3\hat{i} + 2\hat{j} + 6k)$ and $\vec{r} = 7\hat{i} - 6k + \mu(\hat{i} + 2\hat{j} + 2k)$

**Answer:**

$\vec{r} = 2\hat{i} - 5\hat{j} + k + \lambda\left(3\hat{i} - 2\hat{j} + 6k\right)$ ..........(i) and $\vec{r} = 7\hat{i} - 6k + \mu\left(\hat{i} + 2\hat{j} + 2k\right)$ ..........(ii)

Comparing the equations with standard vector from of equation of line

$\vec{r} = \vec{a} + \lambda\vec{b}$ we get $\vec{b_1} = 3\hat{i} - 2\hat{j} + 6k$ and $\vec{b_2} = \frac{3}{4} + 2\hat{j} + 2k$

$\vec{b}_1 \; \vec{b}_2 = 3 + 4 + 12 = 19$

$\left|\vec{b}_1\right| = \sqrt{9+4+36}$ and $\left|\vec{b}_2\right| = \sqrt{1+4+4} = 3$

Angle between the lines (i) and (ii) is given by $\cos^{-1}\left(\frac{\vec{b_1}.\vec{b_2}}{\left|\vec{b_1}\right| . \left|\vec{b_2}\right|}\right) = \cos^{-1}\left(\frac{19}{21}\right)$

**Question: 4** [1]

If A is a 3 x 3 matrix, | A |$\neq$ 0 and | 3A | = k | A |, then write the value of k.

**Answer:**

Given: $|3A| = k.|A|$

We know that $|3A| = 3^n . |A|$, where n is order of A.

Given that order of A is 3.

$\Rightarrow |3A| = 3^3. |A| = k. |A| \Rightarrow k = 27$

**Question: 5** [1]

Let R = {(a, $a^3$) : a is a prime number less than 5} be a relation. Find the range of R.

**Answer:**

Given: R = {(a, $a^3$) : a is a prime number less than 5}.

$\Rightarrow$ R = {(2.2$^3$) , (3,3$^3$)} $\Rightarrow$ Range of R = {8,27}

***Question: 6*** [1]

Write the value of $\cos^{-1}\left(-\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)$

***Answer:***

We have, $\cos^{-1}\left(-\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)$ $[\because \cos(-\theta)=\cos\theta]$

$$\cos^{-1}\left(-\frac{1}{2}\right)=\cos^{-1}\left(\cos\frac{\pi}{3}\right)$$

$$\text{Also } \sin^{-1}\left(-\frac{1}{2}\right)=\sin^{-1}\left(\sin\frac{\pi}{6}\right)$$

$$\therefore \cos^{-1}\left(-\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3}+2\left(\frac{\pi}{6}\right)$$

$$=\frac{\pi}{6} \qquad \left[\because \frac{\pi}{6}\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right]=\frac{\pi}{3}+\frac{\pi}{3}=\frac{2\pi}{3}$$

***Question: 7*** [1]

Write the projection of vector $\hat{i}+\hat{j}+k$ along the vector $\hat{j}$ .

***Answer:***

Required

$$\text{vector}=21\left(\frac{2\hat{i}-3\hat{j}+6k}{\sqrt{22+(-3)2+62}}\right)=21\left(\frac{2\hat{i}-3\hat{j}+6k}{\sqrt{49}}\right)=21\times\frac{2\hat{i}-3\hat{j}+6k}{7}=3\left(2\hat{i}-3\hat{j}+6k\right)=6\hat{i}-9\hat{j}+18k$$

***Question: 8*** [1]

If $\begin{pmatrix} a+4 & 3b \\ 8 & -6 \end{pmatrix}=\begin{pmatrix} 2a+2 & b+2 \\ 8 & a-8b \end{pmatrix}$ , write the value of a-2b.

***Answer:***

Given $\begin{pmatrix} a+4 & 3b \\ 8 & -6 \end{pmatrix}=\begin{pmatrix} 2a+2 & b+2 \\ 8 & a-8b \end{pmatrix}$

On equating, we get a + 4 = 2a + 2, 3b = b + 2, a – 8b = – 6 $\Rightarrow$ a = 2, b = 1.

Now the value of a - 2b = 2 - (2 $\times$ 1) = 2 - 2 = 0

***Question: 9*** [1]

Write the value of the following : $\hat{i}\times\left(\hat{j}+k\right)+\hat{j}\times\left(k+\hat{i}\right)+k\times\left(\hat{i}+\hat{j}\right)$

***Answer:***

$$\hat{i}\times\left(\hat{j}+k\right)+\hat{j}\times\left(k+\hat{i}\right)+k\times\left(\hat{i}+\hat{j}\right)$$

$$=\hat{i}\times\hat{j}+\hat{i}\times k+\hat{j}\times k+\hat{j}\times\hat{i}+k\times\hat{i}+k\times\hat{j}$$

$$=k-\hat{j}+\hat{i}-k+\hat{j}-\hat{i}=\vec{0}$$

***Question: 10*** [1]

Evaluate: $\int \frac{dx}{\sin^2 x \cos^2 x}$

***Answer:***

Let I = $\int \frac{dx}{\sin^2 x \cos^2 x}$

$= \int \operatorname{cosec}^2 x.\sec^2 x\, dx = \int (1+\cot^2 x).\sec^2 x\, dx$

$= \int \sec^2 x\, dx + \int 1+\cot^2 x.\ \sec^2 x\, dx$

$= \tan x + \int \frac{\sec^2 x\, dx}{\tan^3 x}$

$= \tan x + \int z^{-2}\, dz$

$= \tan x + \frac{z^{-2}+1}{-2+1} + c = \tan x - \frac{1}{z} + c$

$= \tan x - \frac{1}{\tan x} + c$

## Section: B

### Question: 11 [4]

Show that the vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar if, and only if $\vec{a}+\vec{b}$ , $\vec{b}+\vec{c}$ and $\vec{c}+\vec{a}$ are coplanar.

### Answer:

- Scalar triple product
- $(\vec{a}\times\vec{b}).\vec{c}=(\vec{b}\times\vec{c})\vec{a}=(\vec{c}\times\vec{a}).\vec{b}=[\vec{a},\vec{b},\vec{c}]=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ = volume of the parallelepiped formed by three non-coplanar vectors $\vec{a}, \vec{b}, \vec{c}$

Step: 1

$\vec{a}, \vec{b}, \vec{c}$ be coplanar, then $[\vec{a},\vec{b},\vec{c}] = 0$

Step: 2

To show that $\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}$ are also coplanar.

Consider $[\vec{a}+\vec{b}\vec{b}+\vec{c}\vec{c}+\vec{a}]$

$=(\vec{a}\times\vec{b}+\vec{a}\times\vec{c}+\vec{b}\times b+\vec{b}\times\vec{c}).(\vec{c}+\vec{a})$

$=(\vec{a}\times\vec{b}+\vec{a}\times\vec{c}+0+\vec{b}\times\vec{c})$

$[\vec{a}\vec{b}\vec{c}]+[\vec{a}\vec{c}\vec{c}]+[\vec{b}\vec{c}\vec{c}]+[\vec{a}\vec{b}\vec{c}]+[\vec{a}\vec{c}\vec{a}]+[\vec{b}\vec{c}\vec{a}]$

$=[\vec{a}\vec{b}\vec{c}]+0+0+0+0+[\vec{a}\vec{b}\vec{c}]$

$2[\vec{a}\vec{b}\vec{c}]=0$

Step: 3

Since $\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}=0$, it follows that they are coplanar.

Step 4

Conversely, if $[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}]$ are coplanar then $[\vec{a}+\vec{b}\vec{b}, +\vec{c}\vec{c}+\vec{a}]=0$

Step: 5

From the above, it can be seen that $[\vec{a}+\vec{b}\vec{b}+\vec{c}\vec{c}+\vec{a}]=2[\vec{a}\vec{b}\vec{c}]$, it follows that $[\vec{a}\vec{b}\vec{c}]=0 \Rightarrow \vec{a},\vec{b},\vec{c}$ are coplanar.

OR

Find a unit vector perpendicular to both of the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ where $\vec{a}=\hat{i}+\hat{j}+k$
$\vec{b}=\hat{i}+2\hat{j}+3k$

***Answer:***

- Unit vector $\perp$ to two vectors and $\vec{a}$ and $\vec{b}$ is given by $\hat{n}=\pm\dfrac{\vec{a}\times\vec{b}}{\left|\vec{a}\times\vec{b}\right|}$
- $\vec{a}\times\vec{b}=\hat{i}\left(a_2b_3-a_3b_2\right)-\hat{j}\left(a_1b_3-a_3b_1\right)+\hat{k}\left(a_1b_2-a2b_1\right)$
- where $\vec{a}=\left(a_1\hat{i}+a_2\,j+a_3\,k\right)$ and $\vec{b}=\left(b_1\hat{i}+b_2\,j+b_3\,k\right)$

*<u>Step:</u> 1*

Let $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+3\hat{k}$

Let us find $\vec{a}+\vec{b}=(\hat{i}+\hat{j}+\hat{k})+(\hat{i}+2\hat{j}+3\hat{k})=2\hat{i}+3\hat{j}+4\hat{k}$

*<u>Step:</u> 2*

Next let us find $\vec{a}-\vec{b}$

$\vec{a}-\vec{b}=(\hat{i}+\hat{j}+\hat{k})-(\hat{i}+2\hat{j}+3\hat{k})=0\hat{i}-1\hat{j}-2\hat{k}=\hat{j}-2\hat{k}$

*<u>Step: 3</u>*

Unit vector $\perp$ to $\vec{a}$ and $\vec{b}$ is given by $\hat{n}=\pm\dfrac{\vec{a}\times\vec{b}}{\left|\vec{a}\times\vec{b}\right|}$. Here we are asked to find a unit vector which is $\perp$ to $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$

$$\hat{n}=\pm\frac{(2\hat{i}+3\hat{j}+4\hat{k})\times(-\hat{j}-2\hat{k})}{\left|(2\hat{i}+3\hat{j}+4\hat{k})\times(-\hat{j}-2\hat{k})\right|}$$

*<u>Step:</u> 4*

Let us determine $(2\hat{i}+3\hat{j}+4\hat{k})\times(-\hat{j}-2\hat{k})$

$$(2\hat{i}+3\hat{j}+4k)\times(-\hat{j}-2k)=\begin{vmatrix} i & j & k \\ 2 & 3 & 4 \\ 0 & -1 & -2 \end{vmatrix}$$

$\hat{i}(6+4)-\hat{j}(-4-0)+k(-2-0)=2\hat{i}+4\hat{j}-2k$

*<u>Step:</u> 5*

$\left|(2\hat{i}+3\hat{j}+4\hat{k})\times(-\hat{j}-2\hat{k})\right| = \sqrt{(-2)^2+4^2+(-2)^2} = \sqrt{4+16+4} = 24$

*Step: 6*

The required unit vector $\hat{n} = \pm\frac{-2\hat{i}+4\hat{j}-2\hat{k}}{\sqrt{24}}$

**Question: 12** [4]

Find the approximate value of f(3.02), upto 2 places of decimal, where $f(x) = 3x^2 + 5x + 3$.

***Answer:***

$$\Rightarrow \begin{bmatrix} 2 & 3 & 4 \\ 4 & 6 & 8 \\ 6 & 9 & 12 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [2\ 3\ 4]$$

$$\begin{bmatrix} 1\times 2 & 1\times 3 & 1\times 4 \\ 2\times 2 & 2\times 3 & 2\times 4 \\ 3\times 2 & 3\times 3 & 3\times 4 \end{bmatrix}$$

The above given matrix has 3 rows and 3 columns. Order of matrix ⇒ 3×3

OR

Find the intervals in which the function $f(x)=\frac{3}{2}x4-4x^3-45x^2+51$ is

i. Strictly increasing
ii. Strictly decreasing

***Answer:***

Here $f(x)=\frac{3}{2}x4-4x^3-45x^2+51$

$\Rightarrow f'(x) = 6x^3 - 12x^2 - 90x \Rightarrow f'(x) = 6x(x^2-2x-15) = 6x\,(x+3)\,(x-5)$

Now for critical point f'(x) = 0

$\Rightarrow 6x(x+3)(x-5) = 0 \Rightarrow x = 0, -3, 5$ i.e. –3, 0, 5 are critical points which divides domain R of given function into four disjoint sub intervals (–∞–3), (–3, 0), (0, 5), (5, ∞)

For (–∞, –3) f'(x) = positive × negative× negative × negative = negative, i.e. f(x) is decreasing in (–∞, –3)

For (– 3, 0) f '(x) = positive - negative + negative × negative = positive, i.e. f(x) is increasing in (–3,0)

For (–0,5) f'(x) = positive × positive × positive × negative = negative, i.e. f(x) is increasing in (–0,5).

For (5, ∞) f'(x) = positive × positive × positive × positive = positive, i.e f(x) is increasing in (5, ∞)

Hence f(x) is (a) strictly increasing in (-3,0) U (5,∞) (b) strictly decreasing in (-∞,-3) U(0,5)

### ***Question: 13*** [4]

Solve the differential equation $(x^2 - yx^2)\,dy + (y^2 + x^2y^2)\,dx = 0$, given that $y = 1$, when $x = 1$.

***Answer:***

The given equation is $(x^2 - yx^2)dy + (y^2 + x^2y^2)dx = 0$

Or, $x^2(1-y)\,dy + y^2(1+x^2)\,dx = 0$

Dividing every term by $x^2y^2$, $\frac{1-y}{y^2}dy + \frac{1+x^2}{x^2}dx = 0$

Integrating both sides, $\int\frac{1-y}{y^2}dy + \int\frac{1+x^2}{x^2}dx = c$

Or, $\int\left(\frac{1}{y^2} - \frac{1}{dy}\right)dy + \int\left(\frac{1}{x^2} - 1\right)dx = c$

Or, $\int\left(y^{-2} - \frac{1}{y}\right)dy + \int\left(x^{-2} + 1\right)dx = c$

Or, $\frac{y^{-1}}{-1} - \log + \frac{x^{-1}}{-1} + x = c$

Or, $x - \left(\frac{1}{x} + \frac{1}{y}\right) - \log y = c$

### ***Question: 14*** [4]

Solve for x: $\cos(\tan^{-1} x) = \sin\,\cot^{-1}\frac{3}{4}$

***Answer:***

$\cos(x) = \sin\left(\frac{\pi}{2} - x\right)$

$\cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x$

Given $\cos(\tan^{-1}x) = \left(\cot^{-1}\frac{3}{4}\right)$

We know that $\cos(x) = \sin\left(\frac{\pi}{2} - x\right)$

$\Rightarrow \sin\frac{\pi}{2} - \tan^{-1}x = \sin -\frac{13}{4}$

$\Rightarrow \frac{\pi}{2} - \tan^{-1}x = \cot^{-1} -\frac{13}{4}$

We know that $\frac{\pi}{2} - \tan^{-1}x = \cot^{-1}x$

$\Rightarrow \cot^{-1}x = \cot\frac{3}{4}$

$\Rightarrow x = \frac{13}{4}$

OR

Prove that : $\cot^{-1} 7 + \cot^{-1} 8 + \cot^{-1} 18 = \cot^{-1} 3$

***Answer:***

$\cot^{-1} x = \tan^{-1}\frac{1}{x}$

$\tan^{-1} x + \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

L.H.S. = $\cot^{-1} 7 + \cos^{-1} 18 = \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} + \tan^{-1}\frac{1}{18}$

$$\tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} = \tan^{-1}\left(\frac{\frac{1}{7}+\frac{1}{8}}{\frac{1-1}{\frac{7.1}{8}}}\right)$$

$$\tan^{-1}\left(\frac{\frac{15}{56}}{\frac{55}{56}}\right) = \tan^{-1}\frac{15}{55} = \tan^{-1}\frac{3}{11}$$

$$\tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} + \tan^{-1}\frac{1}{18}$$

$$\tan^{-1}\frac{3}{11} + \tan^{-1}\frac{1}{18} = \tan^{-1}\left(\frac{\frac{3}{11}+\frac{1}{18}}{\frac{1-3}{\frac{11.1}{18}}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{65}{198}}{\frac{195}{198}}\right)$$

$$= \tan^{-1}\frac{1}{3} = \cot^{-1} = \text{R.H.S}$$

***Question: 15*** [4]

Let f : W→W, be defined as f(x) = x – 1, if x is odd and f(x) = x + 1, if x is even. Show that f is invertible. Find the inverse of f, where W is the set of all whole numbers.

***Answer:***

It is given that:

f: W→W is defined as $\begin{cases} n-1, & \text{if n is odd} \\ n+1, & \text{if n is even} \end{cases}$

One-one:

Let f(n) = f(m). It can be observed that if n is odd, and m is even, then we will have a - 1 = m + 1. ⇒n - m = 2. However, this is impossible. Similarly, the possibility of n being even, and m being odd can also be ignored under a similar argument. Therefore, both n, and m must be either odd or even.

Now, if both n, and m are odd, then we have: f(n) = f(m) ⇒ n−1 = m−1⇒ n =m
Again, if both n, and m are even, then we have: f(n) = f(m) ⇒ n+1 = m+1 ⇒ n = m
Therefore, f is one-one. It is clear that any odd number 2r + 1 in co-domain N is the image of 2r in domain N, and any even number 2r in co-domain N is the image of 2r + 1 in domain N.

Therefore, f is onto. Hence, f is an invertible function.

g: W→W is defined as $g(m)=\begin{cases} m+1, & \text{if } m \text{ is even} \\ m-1, & \text{if } m \text{ is odd} \end{cases}$

Now, when n is odd: gof(n) = g(f(n)) = g(n−1)=n−1+1=n

Note when n is odd, n-1 is even, and, when n is even: gof(n)−g(f(n))=g(n+1)=n+1−1=n

[when n is even, n+1 is odd. Similarly, when m is odd: fog(m) = f(g(m)) = f(m−1) = m−1+1= m

When m is even: fog(m) = f(g(m))= f(m+1) = m+1−1 = m

gof = $I_W$, and fog=$I_W$

Thus, f is invertible and the inverse of f is given by f−1=g, f−1=g, which is the same as f. Hence, the inverse of f is f itself.

***Question: 16*** [4]

Three cards are drawn at random (without replacement) from a well shuffled pack of 52 playing cards. Find the probability distribution of number of red cards. Hence find the mean of the distribution.

***Answer:***

Mean of the probability distribution = $\bar{x}=\Sigma\left(x_i \times p(x_i)\right)$

Three cards are drawn from a pack of 52 cards. Let x be the number of red cards drawn. In the pack of 52 cards there are 26 red cards, and 26 black cards. We can draw 3 cards out of 52 in $^{52}c_3$ ways.

P(x=0)= p (no red cards drawn) = $\dfrac{^{26}c_0 . ^{26}c_3}{^{52}c_3} = \dfrac{2}{17}$

P(x=1)= p (one red cards drawn) = $\dfrac{^{26}c_1 . ^{26}c_2}{^{52}c_3} = \dfrac{13}{34}$

P(x=1)= p (two red cards drawn) = $\dfrac{^{26}c_2 . ^{26}c_1}{^{52}c_3} = \dfrac{13}{34}$

P(x=0)= p (no red cards drawn) = $\dfrac{^{26}c_3 . ^{26}c_0}{^{52}c_3} = \dfrac{2}{17}$

∴ Probability distribution of number of red cards is given by

| $x_i$ | 0 | 1 | 2 | 3 | |
|---|---|---|---|---|---|
| $P(x_i)$ | $\frac{2}{17}$ | $\frac{13}{34}$ | $\frac{13}{34}$ | $\frac{2}{17}$ | |
| $x. P(x_i)$ | 0 | $\frac{13}{34}$ | $\frac{26}{34}$ | $\frac{6}{17}$ | $\sum = x_i.p(x_i)\frac{3}{2}$ |

Mean of the distribution is = $\sum = x_i.p(x_i)\frac{3}{2}$

***Question: 17*** [4]

If x = a cos θ + b sin θ, and y = a sin θ – b cos θ, show that $y^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = 0$

***Answer:***

Given: x = a cosθ+ b sinθ, and y = a sinθ - b cosθ

*Step: 1*

Differentiating y and x with respect to θ, we get $\frac{dx}{d\theta} = -\alpha\sin\theta + \beta\cos\theta$ and

$$\frac{dy}{d\theta} = -\alpha\sin\theta + \beta\cos\theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\alpha\cos\theta + \beta\sin\theta}{\beta\cos\theta - \alpha\sin\theta}$$

*Step: 2*

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right) \cdot \frac{d\theta}{dx}$$

$$\frac{d}{d\theta}\left(\frac{dy}{dx}\right) = \frac{(\beta\cos\theta - \alpha\sin\theta)(-\alpha\sin\theta + \beta\cos\theta) - (\alpha\cos\theta + \beta\sin\theta)(-\beta\sin\theta - \alpha\cos\theta)}{(\beta\cos\theta - \alpha\sin\theta)^2}$$

$$= \frac{(\beta\cos\theta - \alpha\sin\theta)^2 + (\alpha\cos\theta + \beta\sin\theta)^2}{(\beta\cos\theta - \alpha\sin\theta)^2}$$

$$= \frac{\beta^2\cos^2\theta + \alpha^2\sin^2\theta + \alpha^2\cos^2\theta + \beta^2\sin^2\theta)}{(\beta\cos\theta - \alpha\sin\theta)^2}$$

$$\Rightarrow \frac{d}{d\theta}\left(\frac{dy}{dx}\right) = \frac{\alpha^2 + \beta^2}{(\beta\cos\theta + \alpha\sin\theta)^2}$$

*Step: 3*

$$\frac{d\theta}{dx} = \frac{1}{\beta\cos\theta - \alpha\sin\theta}$$

$$\Rightarrow \frac{d}{d\theta}\left(\frac{dy}{dx}\right).\frac{d\theta}{dx} = \frac{\alpha^2 + \beta^2}{(\beta\cos\theta - \alpha\sin\theta)^2} \times \frac{1}{\beta\cos\theta - \alpha\sin\theta} = \frac{\alpha^2 + \beta^2}{(\beta\cos\theta - \alpha\sin\theta)^3}$$

$$\Rightarrow \frac{\alpha^2 + \beta^2}{(\beta\cos\theta - \alpha\cos\theta)^3}$$

*Step: 4*

$$y^2 = (\alpha\sin\theta - \beta\cos\theta)^2$$

$$y^2.\frac{d^2y}{dx^2}=(\alpha\sin\theta-\beta\cos\theta)^2\times\frac{\alpha^2+\beta^2}{(\beta\cos\theta-\alpha\sin\theta)^3}$$

$$=\frac{\alpha^2+\beta^2}{\beta\cos\theta-\alpha\sin\theta}$$

$$x.\frac{dy}{dx}=(\alpha\cos\theta+\beta\sin\theta)\times\frac{\alpha\cos\theta+\beta\sin\theta}{\beta\cos\theta-\alpha\sin\theta}$$

$$=\frac{(\alpha\cos\theta+\beta\sin\theta)^2}{\beta\cos\theta-\alpha\sin\theta}$$

<u>*Step:*</u> *5*

$$y^2\frac{d2y}{dx^2}-x\frac{dy}{dx}+y=\frac{a^2+b^2}{b\cos\theta-a\sin\theta}-\frac{(a\cos\theta-b\sin\theta)}{b\cos\theta-a\sin\theta}+(a\cos\theta-b\sin\theta)$$

$$=\frac{a^2+b^2-\left(a^2\cos^2\theta+b^2\sin^2\theta+b^2\cos^2\theta+a^2\sin^2\theta\right)}{b\cos\theta-a\sin\theta}$$

$$=\frac{a^2+b^2-\left(a^2+b^2\right)}{b\cos\theta-a\sin\theta}=0$$

Hence proved.

***Question: 18*** [4]

Evaluate: $\int\frac{x\cos^{-1}x}{\sqrt{1-x^2}}dx$

***Answer:***

$$I=\int\frac{x\cos^{-1}x}{\sqrt{1-x^2}}dx$$

$$\cos^{-1}x=z\Rightarrow-\frac{1}{\sqrt{1-x^2}}dx=dz\Rightarrow\frac{1}{\sqrt{1-x^2}}dx=-dz$$

$$\therefore I=-\int\cos z.z\,dz$$

$$=-(z.\sin z-\int\sin z\,dz+c$$

$$=-(z.\sin z+\cos z-c)$$

$$=-z\sin z-\cos z+c$$

$$\Rightarrow I=-\cos^{-1}.\sqrt{1-x^2}-x+c$$

$$I=-\sqrt{1-x^2}\cos^{-1}x-x+c$$

***Question: 19*** [4]

Find the distance between the l1 and l2 given by

$l_1:\vec{r}=\hat{i}+2\hat{j}-4k+\lambda(2\hat{i}+3\hat{j}+6k)$

$l_2:\vec{r}=3\hat{i}+3\hat{j}-5k+\mu(2\hat{i}+6\hat{j}+12k)$

***Answer:***

Given lines are

$l_1 : \vec{r} = \hat{i} + 2\hat{j} - 4k + \lambda(2\hat{i} + 3\hat{j} + 6k)$

$l_2 : \vec{r} = 3\hat{i} + 3\hat{j} - 5k + \mu(2\hat{i} + 6\hat{j} + 12k)$

After observation, we get $l_1 \parallel l_2$. Therefore, it is sufficient to find the perpendicular distance of a point of line $l_1$ to line $l_2$. The co-ordinate of a point of $l_1$ is $P(1, 2, -4)$. Also the cartesian form of line $l_2$ is

$$\frac{x-3}{4} = \frac{y-3}{6} = \frac{z+5}{12} \qquad .....(i)$$

Let $Q$(a,b, g) be foot of perpendicular drawn from $P$ to line $l_2$

$\because Q$(a,b, g) lie on line $l_2$

$$\therefore \frac{\alpha-3}{4} = \frac{\beta-3}{6} = \frac{\gamma+5}{12} = \lambda(\text{say})$$

$\Rightarrow \alpha = 4\lambda + 3,\ \beta = 6\lambda + 3,\ \gamma = 12\lambda - 5$

Again, $\overrightarrow{PQ}$ is perpendicular to line $l_2$.

$\because \overrightarrow{PQ}.\vec{b} = 0$, where $\vec{b}$ is parallel vector of $l_2$

$\Rightarrow (\alpha -1).4 + 5(\beta-2).6+(\gamma+4).12 = 0$

$\Rightarrow 4\alpha - 4 + 6\beta - 12 + 12\gamma + 48 = 0$

$\Rightarrow 4\alpha + 6\beta + 12\gamma + 32 = 0$

$\Rightarrow 4(4\lambda + 3) + 6(6\lambda + 3) + 12(12\lambda - 5) + 32 = 0$

$\Rightarrow 16\lambda + 12 + 36\lambda + 18 + 144\lambda - 60 + 32 = 0$

$\Rightarrow 196\lambda + 2 = 0 \qquad \Rightarrow \quad \lambda = \frac{-2}{196} = \frac{-1}{98}$

Co-ordinate of $Q \equiv \left(4\times\left(-\frac{1}{98}\right)+3,\ 6 \times\left(-\frac{1}{98}\right)+3, 12 \times\left(-\frac{1}{98}\right)-5\right)$

$\equiv\left(-\frac{2}{49}+3, -\frac{2}{49}+3. -\frac{6}{49}+5\right) \equiv \left(\frac{145}{49}, \frac{144}{49} - \frac{251}{49}\right)$

Therefore required perpendicular distance is

$$\sqrt{\left(\frac{145}{49}-1\right)^2 + \left(\frac{144}{49}-2\right)^2 + \left(\frac{251}{49}+4\right)^2}$$

$$= \sqrt{\left(\frac{96}{49}\right)^2 + \left(\frac{46}{49}\right)^2 + \left(\frac{-55}{49}\right)^2}$$

$$= \sqrt{\frac{96^2 + 46^2 + 55^2}{49^2}}$$

$$= \sqrt{\frac{9216+2116+3025}{49^2}} = \frac{\sqrt{14357}}{49} = \frac{7\sqrt{293}}{49} = \frac{\sqrt{293}}{7}\text{units}$$

***Question: 20*** [4]

Solve the differential equation $x \log x \frac{dy}{dx} + y = \frac{2}{x} \log x$.

***Answer:***

Dividing both sides by x log x, we can rewrite the given equation as $\frac{dy}{dx} + \frac{y}{\log x} = \frac{2}{x^2}$

The integrating factor is = $e^{\int \frac{dx}{x\log x}} = e^{\log(\log x)} = \log x$

Multiplying the equation by log x we get : logx $\frac{dy}{dx}+\frac{y}{x}=\frac{2}{x^2}\text{logx}$

Integrating, we get : log x = $\int \frac{2}{x^2}\text{logx}\, d\,x + c \Rightarrow \text{ylogx} = 2\int \text{logx} \times x^{-2}dx + c$

Using integration by parts and solving, we get : ylogx = $\frac{-2}{x}$ (1+x) + c

**_Question: 21_** [4]

If cos y = x cos (a + y), where cos a $\neq \pm$ 1, prove that $\frac{dy}{dx}+\frac{\cos^2(a+y)}{\sin a}$

***Answer:***

*<u>Step:</u> 1*

Cos y = x cos(a+y)

$$x = \frac{\cos y}{\cos(a+y)}$$

$$\frac{dy}{dx} = \frac{\cos(a+y).\frac{d}{dy}(\cos y) - \cos y.\frac{d}{dy}(\cos(a+y))}{\cos^2(a+y)}$$

$$= \frac{\cos(a+y).(-\sin y) - \cos y.(-\sin(a+y))}{\cos^2(a+y)}$$

$$= \frac{\sin(a+y)\cos y - \cos(a+y)\sin y}{\cos^2(a+y)}$$

$$= \frac{\sin(a+y-y)}{\cos^2(a+y)}$$

$$= \frac{\sin a}{\cos^2(a+y)}$$

*<u>Step 2 :</u>*

$$\frac{dy}{dx} = \frac{\sin a}{\cos^2(a+y)}$$

$$\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$$

Hence proved.

**_Question: 22_** [4]

Two schools P and Q want to award their selected students on the values of discipline, politeness, and punctuality. The school P wants to award ₹x each, ₹y each, and ₹x each, for the respective values to its 3, 2, and 1 students with a total award money of ₹1000.

School Q wants to spend ₹1500 to award its 4, 1, and 3 students on the respective values (by giving the same award money for the three values before). If the total amount of awards for one prize on each value is ₹600, using matrices, find the award money for each value. Apart from the above three values, suggest one more value for awards.

***Answer:***

Here we can say,

3x + 2y + z = 1000

4x + y + 3z = 1500

x + y + z = 600

Therefore,

$$\begin{pmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1000 \\ 1500 \\ 600 \end{pmatrix}, \text{ or}$$

A . X = B

$|A| = 3(-2) - 2(1) + 1(3) = -5 \neq 0$

$\therefore X = A^{-1} B$

Co – factor's are,

$A_{11} = -2,\quad A_{12} = -1,\quad A_{13} = 3$

$A_{21} = -1,\quad A_{22} = 2,\quad A_{23} = -1$

$A_{31} = 5,\quad A_{32} = -5,\quad A_{33} = -5$

Hence,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{5}\begin{pmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{pmatrix}\begin{pmatrix} 1000 \\ 1500 \\ 600 \end{pmatrix}$$

$\therefore$ x = 100 , y = 200 , z = 300

i.e., ₹100 for discipline, ₹ 200 for politeness, and ₹ 300 for punctuality. One more value like sincerity, truthfulness, etc.

## Section: C

**Question: 23** [6]

A cottage industry manufactures pedestal lamps and wooden shades, each requiring the use of a grinding/cutting machine, and a sprayer. It takes 2 hours on the grinding/cutting machine, and 3 hours on the sprayer to manufacture a pedestal lamp. It takes 1 hour on the grinding/cutting machine, and 2 hours on the sprayer to manufacture a shade. On any day, the sprayer is available for at the most 20 hours, and the grinding/cutting machine for at the most 12 hours. The profit from the sale of a lamp is < 25, and that from a shade is < 15. Assuming that the manufacturer can sell all the lamps, and shades that he produces, how should he schedule his daily production in order to maximise his profit. Formulate an LPP and solve it graphically.

**Answer:**

Let x be the number of pedestal lamps and y be the number of wooden shades that we can make. Our problem is to maximize x and y and the maximum profit at a given capacity. Clearly, $x, y \geq 0$. Let us construct the following table from the given data:

| | *Pedestal Lamps (x)* | *Wooden Shades (y)* | *Time available* |
|---|---|---|---|
| Grinding Machine (h) | 2 | 1 | 12h |
| Sprayer (h) | 3 | 2 | 20h |
| Profits (Rs.) | 5 | 3 | |

We have the following constraints: $2x+y \leq\leq 12$, and $3x+2y \leq\leq 20$

The profit on pedestal lamps is Rs. 5, and on wooden lampshades is Rs. 3. We need to maximize the profits, i.e. maximize $5x + 3y$, given the above constraints.

*Plotting the constraints:*

Plot the straight lines $2x + y = 12$ and $3x + 2y = 20$. First draw the graph of the line $2x + y = 12$

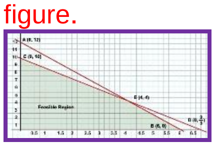
If $x = 0$, $y = 12$, and if $y = 0$, $x = 6$. So, this is a straight line between (0,12), and (6,0). At (0,0), in the inequality, we have $0 + 0 = 0$, which is $\leq 0$. So the area associated with this inequality is bounded towards the origin. Similarly, draw the graph of the line $3x + 2y = 20$.

If $x = 0$, $y = 10$, and if $y = 0$, $x = \frac{20}{3}$. So, this is a straight line between (0,10), and $\left(\frac{20}{3}, 0\right)$. At (0,0), in the inequality, we have $0 + 0 = 0$, which is $\leq\leq 0$. So the area associated with this inequality is bounded towards the origin. Finding the feasible region: We can see that the feasible region is bounded and in the first quadrant. On solving the equations $2x + y = 12$, and $3x + 2y = 20$, we get,

$3x + 2(12-2x) = 20 \rightarrow 3x + 24 - 4x = 20 \rightarrow x = 4.$

If $x = 4$, $y = 12 - 2 \times 4 = 4$. $\Rightarrow x=4, y=4$

Therefore the feasible region has the corner points (0,0), (0,10), (4,4), (6,0) as shown in the figure.



*Solving the objective function using the corner point method*

The values of Z at the corner points are calculated as follows:

| Corner Point | Z = 5x+3y |
|---|---|
| O (0,0) | 0 |
| C (0,10) | 30 |
| E (4,4) | 32 (Max Value) |
| B (6.0) | 30 |

The maximum profit we can make is Rs. 32, which involves making 4 pedestal lamps and 4 wooden shades.

***Question: 24*** [6]

Find the equation of the plane that contains the point (1, – 1, 2) and is perpendicular to both the planes 2x + 3y – 2z = 5, and x + 2y – 3z = 8. Hence find the distance of point P(–2, 5, 5) from the plane obtained above.

***Answer:***

Distance of the point $(x_1,y_1, z_1)$ from the plane ax + by + cz + d = o is given by $\left|\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right|$ To form the equation of a plane we need a point on it and normal vector ( $\vec{n}$ )

*Step 1:*

Given that the required plane contains the point (1, -1,2). Let the normal to the required plane be $\vec{n}$ . Also it is given that the required plane is $\perp$ to the planes.

2x + 3y - 2z – 5 = 0…………(i), and x + 2y – 3z – 8……. (ii)

Normal to the plane is $\overrightarrow{n_1}$ = (2,3, -2). Normal to the plane (ii) is $\overrightarrow{n_2}$ = (1,2, -3)

Since the required plane is $\perp$ to (i) and (ii)

$$\vec{n}=\overrightarrow{n_1}\times\overrightarrow{n_2}$$

$$\vec{n}=\begin{vmatrix} i & \hat{j} & k \\ 2 & 3 & -2 \\ 1 & 2 & -3 \end{vmatrix}\hat{i}(-9+4)-\hat{j}(-6+2)+k(4-3)$$

i.e., $\vec{n}$ = ( -5,4,1)

*Step 2:*

$\therefore$ Equation of the required plane is -5x + 4y + z + d = 0

Given that this plane contains the point (1, -1, 2)

$\Rightarrow$ it satisfies the equation of the plane. $\Rightarrow$ -5 × 1 + 4 × (-1) + 2 +d = 0 $\Rightarrow$ d = 7

$\Rightarrow$ Equation of the required plane is 5x – 4y – z – 7 = 0

*Step 3:*

We know that distance of the point $(x_1,y_1, z_1)$ from the plane ax + by + cz + d = 0 is given by

$$\left|\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right|$$

The distance of the point (-2, 5, 5) from the plane 5x - 4y – z - 7 = 0 is given by

$$\left|\frac{5\times(-2)+(-4)\times5+(-1)\times5-7}{\sqrt{5^2+(-4)^2+(-1)^2}}\right| = \left|\frac{-10-20-5-7}{\sqrt{42}}\right| = \frac{42}{\sqrt{42}} = \sqrt{42}$$

OR

Find the distance of the point P(–1, –5, –10), from the point of intersection of the line joining the points A(2, –1, 2), and B(5, 3, 4) with the plane x – y + z = 5.

***Answer:***

Given: A (2, -1, 2), B (5,3,4). Direction ratio of the line AB = (5 -2,3+1, 4-2) = (3,4,2). Equation of the line AB is given by

$$\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{2}=\lambda \ldots\ldots\ldots(i)$$

Any point on this line is given in terms of $\lambda$ by $c(3\lambda+2, 4\lambda-1, 2\lambda+2)$

Let the line AB intersect the plane x-y+z = 5 at this point C. $\Rightarrow$ C lies on the plane too.

$\therefore$ C satisfies the equation of the plane. $\Rightarrow (3\lambda+2)-(4\lambda-1)+(2\lambda+2)=5 \Rightarrow \lambda=0$

$\therefore$ C (2, -1, 2) We have to find the distance of the point P (-1, -5, -10) from C (2, -1, 2), which is given by $\sqrt{(2+1)^2+(-1+5)^2+(2+10)^2}=\sqrt{9+16+144}=13$

***Question: 25*** [6]

Two schools P and Q want to award their selected students on the values of Tolerance, Kindness, and Leadership. The school P wants to award ₹ x each, Rs. y each, and ₹ z each for the three respective values to 3, 2, and 1 students respectively with a total award money of ₹ 2,200. School Q wants to spend ₹ 3,100 to award its 4, 1, and 3 students on the respective values (by giving the same award money to the three values as school P). If the total amount of award for one prize on each value is ₹1,200, using matrices, find the award money for each value. Apart from these three values, suggest one more value which should be considered for award.

***Answer:***

3 equations are formed from the given statements, i.e., Given: 3x+2y+z=2200
4x+y+3z=3100, and x+y+z=1200

Converting the system of equations in matrix form we get, $\begin{bmatrix}3&2&1\\4&1&3\\1&1&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2200\\3100\\1200\end{bmatrix}$, i.e., AX=B

where $A=\begin{bmatrix}3&2&1\\4&1&3\\1&1&1\end{bmatrix}$, $X=\begin{bmatrix}x\\y\\z\end{bmatrix}$ and, $B=\begin{bmatrix}2200\\3100\\1200\end{bmatrix} \Rightarrow X=A^{-1}.B$

$$A^{-1}=\frac{1}{|A|}(AdjA)$$

|A|=3(1−3) −2(4−3) +1(4−1) =−6−2+3=−5

$$AdjA=\begin{bmatrix}-2&-1&5\\-1&2&-5\\3&-1&-5\end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{-5}\begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 2 & 1 & -5 \\ 1 & -2 & 5 \\ -3 & 1 & 5 \end{bmatrix} \Rightarrow X = \frac{1}{5}\begin{bmatrix} 2 & 1 & -5 \\ 1 & -2 & 5 \\ -3 & 1 & 5 \end{bmatrix}\begin{bmatrix} 2200 \\ 3100 \\ 1200 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 4400+3100-6000 \\ 2200-6200+6000 \\ -6600+3100+6000 \end{bmatrix} = \begin{bmatrix} 300 \\ 400 \\ 500 \end{bmatrix}$$

$\Rightarrow$ x =300, y = 400 and z = 500, i.e., the award money for each value are ₹ 300 for Tolerance, ₹400 for Kindness and, ₹ 500 for Kindness.

***Question: 26*** [6]

An insurance company insured 2000 scooter drivers, 4000 car drivers, and 6000 truck drivers. The probabilities of an accident for them are 0.01, 0.03, and 0.15 respectively. One of the insured persons meets with an accident. What is the probability that he is a scooter driver or a car driver ?

***Answer:***

Given the total number of drivers are as follows: 2000 scooter drivers, 4000 car drivers, and 6000 truck drivers. Total = 2000+4000+6000 = 12000.

Let $E_1$ be the event that the insured person is a scooter driver.

$$P(E_1) = \frac{\text{Number of scooter drivers}}{\text{Total number of drivers}} = \frac{2000}{12000} = \frac{1}{6} = \frac{2000}{12000} = \frac{1}{6}$$

Let $E_2$ be the event that the insured person is a car driver.

$$P(E_2) = \frac{\text{Number of scooter drivers}}{\text{Total number of drivers}} = \frac{4000}{12000} = \frac{1}{3}$$

Let $E_3$ be the event that the insured person is a truck driver.

$$P(E_3) = \frac{\text{Number of scooter drivers}}{\text{Total number of drivers}} = \frac{6000}{12000} = \frac{1}{2} = \frac{6000}{12000} = \frac{1}{2}$$

Let A: event that the insured person met w an accident.

P (scooter driver met w/ an accident) = $P(A|E_1) = 0.01 = \frac{1}{100}$

P (car driver met w/ an accident) = $P\left(\frac{A}{E_1}\right) = 0.03 = \frac{3}{100}$

P (truck driver met w/ an accident) = $P\left(\frac{A}{E_2}\right) = 0.15 = \frac{15}{100}$

The probability that a driver is a scooter driver who met w/ an accident is given by $P\left(\frac{E_1}{A}\right)$

We can use Baye's theorem, according to which P ($E_1$A )

$$= \frac{P(E_1)(P(A|E_1)}{P(E_1)P(A|E_1)+P(E_2)(A|E_2)+P(E_3)P(A|E_3)}$$

$$P\left(\frac{E_1}{A}\right) = \frac{\frac{1}{6}\cdot\frac{1}{100}}{\frac{1}{6}\cdot\frac{1}{100}+\frac{1}{3}\cdot\frac{3}{100}+\frac{1}{2}\cdot\frac{15}{100}} = \frac{1}{1+6+45} = \frac{1}{52}$$

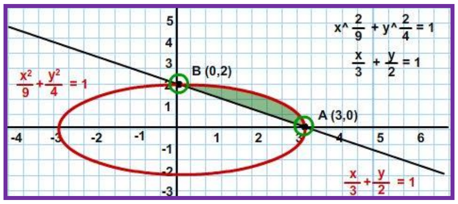
Extra: We can easily show that P (that he is a car driver) = $\frac{6}{52}$

***Question: 27*** [6]

Find the area of the smaller region bounded by the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ , and the line $\frac{x}{3}+\frac{y}{2}=1$.

***Answer:***

Hence the required area is the area enclosed between the straight line, and the ellipse. To find the limits, the semi major axis extends from 0 to 3, and the x intercept of the straight line is also 3. So the limits are from 0 to 3.




The required area is $A=\int_a^b [f(x)-g(x)]dx$

Here a=0, b=3,f(x)= $\frac{x^2}{9}+y^2=1$

$==\frac{2}{3}\left[\sqrt{9-x^2}\right]$

g(x)= $\frac{x}{3}+\frac{y}{2}=1=\frac{2}{3}(3-x).$

A= $\frac{2}{3}\int_0^3\sqrt{9-x^2}dx-\frac{2}{3}\int_0^3(3-x)dx$

On integrating we get,

$$A=\frac{2}{3}\left[\frac{x}{2}\sqrt{9-x^2}+\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_0^3=\frac{2}{3}\left[3x-\frac{x^2}{2}\right]_0^3$$

On applying the limits we get,

$$A=\frac{2}{3}\left[\frac{9}{2}\left(\frac{\pi}{2}\right)\right]-\frac{2}{3}\left[9-\frac{9}{2}\right]=\frac{2}{3}\left[\frac{9\pi}{4}-\frac{9}{2}\right]=\frac{2}{3}\times\frac{9}{4}(\pi-2)$$

Hence the required area is 32(π−2)sq.units

***Question: 28*** [6]

The sum of the perimeters of a circle, and a square is k, where k is some constant. Prove that the sum of their areas is least when the side of the square is equal to the diameter of the circle.

***Answer:***

*Step 1:*

Let x = Radius of the circle, y= Side of the square.
Circumference of the circler = 2πx = 2πr, Perimeter of square=4s=4y
Sum of perimeters of circle and square=2πx+4y=k----(1)
Area of circle $=\pi r^2=\pi x2=\pi r^2$, Area of square=$y^2$(i.e$s^2$)

*Step 2:*

Sum of areas of circle, and square = $\pi x^2+y^2$

$A = \pi x^2 + y^2$----(2)
From (1) $2\pi x + 4y = k$, $4y = k - 2\pi x$

$y = \frac{k - 2\pi x}{4}$ -----(3)

$A = \pi x^2 + \left[\frac{k - 2\pi x}{4}\right]^2$ [By substituting the value of y]

*Step 3:*

Differentiating with respect to x we get, $\frac{dA}{dx} = 2\pi x + 2\left[\frac{k - 2\pi x}{4}\right]\left[\frac{-2\pi}{4}\right] = 2\pi x - \frac{\pi}{4}[k - 2\pi x]$

$= \left[2\pi + \frac{\pi^2}{2}\right] \times \frac{k\pi}{4}$

$\frac{dA}{dx} = 0$ at

$X = \frac{k\pi}{4} \times \frac{2}{4\pi + \pi^2} = \frac{k}{2(\pi + 4)}$

On double differentiation we get $\frac{d^2A}{dx^2} = [2\pi + \frac{\pi^2}{2}] \Rightarrow$ positive

A is least when $x = \frac{k}{2(\pi + 4)}$

*Step 4:*

From (3)

$y = \frac{1}{4}[k - 2\pi . \frac{k}{2(\pi + 4)}]$

$= \frac{k}{4}\left[\frac{\pi + 4 - \pi\pi + 4}{\pi + 4}\right]$

$= \frac{k}{\pi + 4}$

*Step 5:*

$\frac{y}{x} = \frac{\frac{k}{\pi + 4}}{\frac{k}{2(\pi + 4)}} = \frac{k}{\pi + 4} \times \frac{2(\pi + 4)}{k} = 2$

Therefore, $y = 2x \rightarrow$ proves that the sum of their areas is least when the side of square (y) = diameter, i.e, 2 times the radius (x)

### ***Question: 29*** [6]

Find the distance of the point (-1, -5, -10) from the point of intersection of the line $\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda\left(3\hat{i} + 4\hat{j} + 2\hat{k}\right)$, and the plane $\vec{r}.\left(\hat{i} - \hat{j} - \hat{k}\right) = 5$

### ***Answer:***

General point of the line is $(2+3\lambda)\hat{i} + (-1+4\lambda)\hat{j} + (2+2\lambda)\hat{k}$

Putting in the equation of plane; we get $1.(2+3\lambda) - 1.(-1+4\lambda) + 1.(2+2\lambda) = 5$

$\therefore \lambda = 0$

Point of intersection: $2\hat{i} - \hat{j} + 2\hat{k}$ or (2, -1, 2)

Distance: $\sqrt{(2+1)^2 + (-1+5)^2 + (2+10)^2} = \sqrt{169}$ = 13