2014

Section A *(Question numbers 1 to 10 carry 1 mark each)*

Question: 1

Let * be a operation, on the set of all non – zero real numbers, given by a * b = $\frac{ab}{5}$ for all a, b $\in$ R - $\{0\}$. Find the value of x, given that $2 * (x * 5) = 10$.

Answer:

Given, $a * b = \left(\frac{ab}{5}\right), \forall a, b \in R - \{0\}$ (i)

Also, 2 * (x * 5) = 10 (ii)

From (i)

2*(x* 5) = 10, or

$2 * \left(\frac{x.5}{5}\right) = 10$, or

2*x = 10, or

$\left(\frac{2x}{5}\right) = 10$, or

x = 25

Question: 2

If $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1}x\right) = 1$, then find the value of x.

Answer:

Given $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1}x\right) = 1$

When, $\sin\theta = x \Rightarrow \theta = \sin^{-1}x$, then $\sin^{-1}\frac{1}{5} + \cos^{-1}x = \sin^{-1}(1)$

Also, because $\sin\frac{\pi}{2} = 1$, therefore, $\sin^{-1}\frac{1}{5} + \cos^{-1}x = \sin^{-1}\left(\sin\frac{\pi}{2}\right)$

But we also know that, $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$, $x \in [-1, 1]$, hence

$\sin^{-1}\frac{1}{5} + \cos^{-1}x = \frac{\pi}{2}$, or

$\sin^{-1}\frac{1}{5} = \sin^{-1}x$, or

$x = \frac{1}{5}$

Question: 3

If $2\begin{bmatrix} 3 & 4 \\ 5 & x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$, find (x - y).

Answer:
Given,

$$2\begin{bmatrix} 3 & 4 \\ 5 & x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}, \text{ or}$$

$$\begin{bmatrix} 6 & 8 \\ 10 & 2x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}, \text{ or}$$

$$\begin{bmatrix} 7 & 8+y \\ 10 & 2x+1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$$

On comparing the corresponding elements, we get
$8 + y = 0$
$\therefore\ y = -8$

And,
$2x + 1 = 5$

$$\therefore x = \frac{5-1}{2} = 2$$

$x - y = 2 - (-8) = 10$

Question: 4

Solve the following matrix equation for x: $\begin{bmatrix} x & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} = 0$

Answer:

We have $\begin{bmatrix} x & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} = 0$

By using matrix multiplication we get
$\begin{bmatrix} x-2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$

On comparing the corresponding elements from both sides, we get
$x - 2 = 0$
$\therefore\ x = -2$

Question: 5

If $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$, write the value of x.

Answer:
We have,

$$\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}, \text{ or}$$

$2x^2 - 40 = 18 - (-14)$, or

$2x^2 - 40 = 32$, or

$2x^2 = 72$, or

$x = \pm 6$

Question: 6

Write the antiderivative of $\left(3\sqrt{x}+\frac{1}{\sqrt{x}}\right)$.

Answer:

Antiderivative of $\left(3\sqrt{x}+\frac{1}{\sqrt{x}}\right)$ is

$$\int\left(3\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx = \left(3\int\sqrt{x}dx+\int\frac{1}{\sqrt{x}}dx\right) = 3\int\left(\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}-1}\right)+\left(\frac{x^{-\frac{1}{2}-1}}{-\frac{1}{2}-1}\right) = 2x^{\frac{3}{2}}+x$$

Question: 7

Evaluate: $\int_0^3\left(\frac{dx}{9+x^2}\right)$

Answer:

Let

$$L = \int_0^3\frac{dx}{9+x^2} = \int_0^3\frac{dx}{x^2+(3)^2} = \left[\frac{1}{3}\tan^{-1}\frac{x}{3}\right]_0^3 \text{ when, } \int\left(\frac{dx}{x^2+a^2}\right) = \left\{\frac{1}{a}\times\tan^{-1}\left(\frac{x}{a}\right)\right\}, \text{ or}$$

$$L = \frac{1}{3}\left[\tan^{-1}\left(\frac{3}{3}\right)-\tan^{-1}(0)\right] = \frac{1}{3}\left[\tan^{-1}(1)-0\right] = \frac{1}{3}\left(\frac{\pi}{4}\right) = \frac{\pi}{12}$$

Question: 8

Find the projection of the vector $\hat{i}+3\hat{j}+7\hat{k}$ on the vector $2\hat{i}+3\hat{j}+6\hat{k}$.

Answer:

Let, a = $\hat{i}+3\hat{j}+7\hat{k}$, and b = $2\hat{i}+3\hat{j}+6\hat{k}$. The projection of vector ‘a’ on the vector ‘b’ is given by

$$\frac{1}{|\vec{b}|}\times\left(\vec{a}.\vec{b}\right) = \left\{\frac{(1\times 2)-(3\times 3)+(7\times 6)}{\sqrt{(2)^2+(-3)^2+(6)^2}}\right\} = \left(\frac{2-9+42}{\sqrt{4+9+36}}\right) = \frac{35}{\sqrt{49}} = \frac{35}{7} = 5$$

Question: 9

Find the values of a, b, c, and d from the equations, $\begin{bmatrix}2a+b & a-2b\\ 5c-d & 4c+3d\end{bmatrix} = \begin{bmatrix}4 & -3\\ 11 & 24\end{bmatrix}$

Answer:

By equality of two matrices, we have
2a+b = 4, or
a-2b =-3, or
5c-d =11, or
4c+3d = 24

On solving the above equation’s, we get a=1, b=2, c=3, and d=4.

Question: 10

Find the principal values of thn $^{-1}$ (-1).

Answer:

Let, $x = \text{thn}^{-1}(-1)$

$\text{Thn}\, x = -1 = -\text{thn}\left(\pi - \frac{\pi}{4}\right)$, or

$Q\, \text{thn}(\pi - \theta) = \text{thn}\theta$, or

$\text{thnx} = \text{thn}\left(\frac{3\pi}{4}\right)$, or

$x = \left(\frac{3\pi}{4}\right)$

$\therefore$ the principal value of thn^{-1}(-1) is $\left(\frac{3\pi}{4}\right)$

Section: B *(Question numbers 11 to 22 carry 4 mark each)*

Question: 11

Let A = $\{1, 2, 3, \ldots\ldots\ldots, 9\}$ and R be the relation in $A \times A$ defined by (a, b) R (c, d) if a + d = b + c for (a, b), (c, d) in $A\times$ A. Prove that R is an equivalence relation. Also obtain the equivalence class $(2,5)$.

Answer:

Given, relation R defined by (a, b) R (c, d), if a + d = b + c for (a, b), (c, d) in A $\times$ A

Here, A = $\{1,2,3,\ldots\ldots\ldots,9\}$

We observe the following properties on 'R' Reflexive. Let (1, 2) be an element of $A\times$A.

Then,

$(1, 2) \in A\times A$, or

$1,2 \in A$

$1+2=2+1$ [$\because$ Addition is commutative]

(1, 2) R (1, 2) for all (1, 2) $\in$ A $\times$ d

So, R is reflexive on $A\times A$

Symmetric let (1, 2), (3, 4) $\in$ A $\times$ A such that (1, 2) K (3, 4). Then, 1+4 = 2+3

3+2 = 4+1 [$\because$ Addition is commutative]

(3, 4) R (1, 2)

Thus (1, 2) R (3, 4) $\Rightarrow$ (3, 4) R (1, 2) $\in$ A $\times$ A

So, R is symmetric on $A\times$ A.

Transitive Let (1, 2), (3, 4), (5, 6) $\in$ $A\times$A such that (1, 2) R (3, 4), and (3, 4) R (5, 6). Then

(1, 2) R (3, 4) $\Rightarrow$ 1+4 = 2 + 3

(3, 4) R (5, 6) $\Rightarrow$ 3 + 6 = 4 + 5, or

(1 + 4) + (3 + 6) = (2 + 3) + (4 + 5), or

1 + 6 = 2 + 5, or

(1, 2) R (5, 6)

Thus (1, 2) R (3, 4), and (3, 4) R (5, 6), or

(1, 2) R (5, 6) for all (1, 2), (3, 4), (5, 6) $\in$ A $\times$ A

So, R is transitive on $A\times A$.

Hence, it is an equivalence relation on $A\times A$ equivalence class containing an element x of A given by $[X]_R = \{y|(x,y)\in R\}$

Here, equivalence class: [(2, 5)] = $\{(1,4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)\}$

Question: 12

Prove that $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)=\frac{x}{2}$; $\quad x\in\left(0,\frac{\pi}{4}\right)$

Answer:

From L.H.S we start decomposition as below,

$$= \cot^{-1}\left[\left\{\frac{\left(\sqrt{1+\sin x}+\sqrt{1-\text{Sin}x}\right)}{\left(\sqrt{1+\sin x}-\sqrt{1-\sin x}\right)}\right\}\times\left\{\frac{\left(\sqrt{1+\sin x}+\sqrt{1-\text{Sin}x}\right)}{\left(\sqrt{1+\sin x}+\sqrt{1-\text{Sin}x}\right)}\right\}\right] \qquad \because 1=\left\{\frac{\left(\sqrt{1+\sin x}+\sqrt{1-\text{Sin}x}\right)}{\left(\sqrt{1+\sin x}+\sqrt{1-\text{Sin}x}\right)}\right\}$$

$$= \cot^{-1}\left[\frac{\left(\sqrt{1+\sin x}+\sqrt{1-\text{Sin}x}\right)^2}{\left\{\left(\sqrt{1+\sin x}\right)^2-\left(\sqrt{1-\text{Sin}x}\right)^2\right\}}\right] \qquad \left\{\because (a+b)(a-b)=a^2-b^2\right\}$$

$$= \cot^{-1}\left[\frac{\left\{(1+\sin x)+(1-\sin x)+\left(2\sqrt{1-\sin^2 x}\right)\right\}}{(1+\sin x-1+\sin x)}\right] \qquad \left\{\because (a+b)^2=a^2+b^2+2ab\right\}$$

$$= \cot^{-1}\left(\frac{2+2\cos x}{2\sin x}\right) \qquad \left\{\because 1-\sin^2 x=\cos^2 x\right\}$$

$$= \cot^{-1}\left(\frac{1+\cos x}{\sin x}\right)$$

$$= \cot^{-1}\left(\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right) \qquad \left(\text{when, } \cos x = 2\cos^2\frac{x}{2}-1, \text{ and} \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}\right)$$

$$= \cot^{-1}\left(\cot\frac{x}{2}\right) = \frac{x}{2} \text{ (R.H.S)}$$

OR

Prove that $2\tan^{-1}\left(\frac{1}{5}\right)+\sec^{-1}\left(\frac{5\sqrt{2}}{7}\right)+2\tan^{-1}\left(\frac{1}{8}\right)=\frac{\pi}{4}$

Answer:

From, L.H.S we get

$$2\tan^{-1}\left(\frac{1}{5}\right)+\sec^{-1}\left(\frac{5\sqrt{2}}{7}\right)+2\tan^{-1}\left(\frac{1}{8}\right)$$

$$= 2\left\{\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}\right\}+\sec^{-1}\frac{5\sqrt{2}}{7}$$

$$=2\tan^{-1}\left\{\frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}\times\frac{1}{8}}\right\}+\tan^{-1}\sqrt{\left(\frac{5\sqrt{2}}{7}\right)^2-1} \qquad \left[\begin{array}{l}\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left\{\frac{x+y}{1-xy}\right\} \\ \because \text{ and } \sec^{-1}x=\tan^{-1}\sqrt{x^2-1}\end{array}\right]$$

$$= 2\tan^{-1}\frac{13}{39}+\tan^{-1}\sqrt{\frac{50}{49}-1}$$

$$=2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\left\{\frac{2\times\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2}\right\}+\tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{7} = \tan^{-1}\left\{\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\times\frac{1}{7}}\right\}$$

$$= \tan^{-1}(1) \; = \frac{\pi}{4} \text{ (R.H.S)}$$

Question: 13
Using property of determinants, prove that

$$\begin{vmatrix} 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix} = (x+y+z)^3$$

Answer:
From L.H.S we have

$$= \begin{vmatrix} 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix} \qquad \text{(Applying } R_z \rightarrow R_1 + R_2 + R_3 \text{ we get)}$$

$$= \begin{vmatrix} x+y+z & x+y+z & 1 \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix} \qquad \text{\{Taking (x+y+z) common from } R_1 \text{ we get\}}$$

$$= (x+y+z) \times \begin{vmatrix} 1 & 1 & 1 \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix} \qquad \left\{\text{as}, \begin{pmatrix} C_2 \rightarrow C_2 - C_1 \\ C_3 \rightarrow C_3 - C_1 \end{pmatrix}\right\}$$

$$= (x+y+z) \times \left\{0 \times (x+y+z) + (x+y+z)^2\right\}$$

$$= (x+y+z)^3$$

Question: 14

Differentiate $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ with respect to $\cos^{-1}\left(2x\sqrt{1-x^2}\right)$, when $x \neq 0$.

Answer:

Let, $u = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

Putting, $x = \cos\theta \Rightarrow \theta = \cos^{-1} x$

$$u = \tan^{-1}\left(\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}\right) = \tan^{-1}\left(\frac{\sqrt{\sin^2\theta}}{\cos\theta}\right) = \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right) = \tan^{-1}(\tan\theta) \; = \theta$$

Therefore,

$u = \cos^{-1} x \qquad [\because x = \cos\theta]$

On differentiating w.r.t. x we get

$$\frac{du}{dx} = -\left(\frac{1}{\sqrt{1-x^2}}\right)$$

Again let,

$$v = \cos^{-1}\left(2x\sqrt{1-x^2}\right)$$

Putting $x = \cos\theta \Rightarrow \theta = \cos^{-1}x$

$$v = \cos^{-1}\left(2\cos\theta\sqrt{1-\cos^2\theta}\right)$$

Since, $\sin^2\theta + \cos^2\theta = 1 \Rightarrow \sin\theta = \sqrt{1-\cos^2\theta}$, we get

$$v = \cos^{-1}(2\cos\theta\sin\theta) = \cos^{-1}(\sin 2\theta) = \cos^{-1}\left\{\cos\left(\frac{\pi}{2} - 2\theta\right)\right\} = \frac{\pi}{2} - 2\theta = \frac{\pi}{2} - 2\cos^{-1}x$$

On differentiating w.r.t. x, we get

$$\frac{dv}{dx} = \left(\frac{2}{\sqrt{1-x^2}}\right)$$

Now,

$$\frac{dv}{dx} = \left(\frac{du}{dx} \times \frac{dx}{dv}\right) = \left\{-\left(\frac{1}{\sqrt{1-x^2}}\right) \times \left(\frac{\sqrt{1-x^2}}{2}\right)\right\} = -\frac{1}{2}$$

Question: 15

If $y - x^x$, then prove that $\frac{d^2y}{dx^2} - \left\{\frac{1}{y} \times \left(\frac{dy}{dx}\right)^2\right\} - \frac{y}{x} = 0$

Answer:

We have, $y = x^x$

Taking log on both sides, we get

$\log y = \log x^x$, or

$\log y = x\log x$

On differentiating both sides w.r.t. x we get

$$\frac{1}{y} \times \left(\frac{dy}{dx}\right) = x\frac{d}{dx}(\log x) + \log x\frac{d}{dx}(x) = x\left(\frac{1}{x}\right) + \log x \text{, or}$$

$$\frac{dy}{dx} = y(1 + \log x) = y(1 + \log x) \qquad (1)$$

Again differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = y\frac{d}{dx}(1+\log x) + (1+\log x)\frac{dy}{dx} = \left\{y\left(\frac{1}{x}\right)(1+\log x)\left(\frac{dy}{dx}\right)\right\} = \left\{\frac{y}{x} + (1+\log x)\frac{dy}{dx}\right\} \qquad (2)$$

Now, from L.H.S.

$$\frac{d^2y}{dx^2}-\frac{1}{y}\left(\frac{dy}{dx}\right)^2-\frac{y}{x}=\left(\frac{y}{x}\right)+\left\{(1+\log x)\frac{dy}{dx}\right\}-\left[\frac{1}{y}\{y(1+\log x)\}^2\right]-\left(\frac{y}{x}\right)$$

From (1), and (2)

$$=\left(\frac{y}{x}\right)+\left[(1+\log x)\times\{y(1+\log x)\}\right]-\left[\frac{y}{x}\times\left\{y^2(1+\log x)^2\right\}\right]-\left(\frac{y}{x}\right)$$

$$=\left\{\frac{y}{x}(1+\log x)^2\right\}-\left\{y(1+\log x)^2\right\}=0 \text{ (R.H.S)}$$

Question: 16

Find the intervals in which the function f(x) = $3x^4+4x^3-12x^2+5$ is

Answer:

The given function is $f(x)=3x^4-4x^3-12x^2+5$

On differentiating w.r.t. x, we get, $f'(x)=12x^3-12x^2+5$

On putting f(x) = 0, we get

$12x^3-12x^2-24x=0$, or

$12x\left(x^2-x-2\right)=0$, or

$12x\left(x^2-2x+x-2\right)=0$, or

$12x(x+1)(x-2)=0$, or

$x=0,-1,\text{or } 2$

Now, we find intervals in which f(x) is strictly increasing and the intervals in which it is strictly decreasing.

Interval	$f'(x)=12x(x+1)(x-2)$	**Sign of** $f'(x)$
$x<-1$	$(-)\ (-)\ (-)$	-ve
$-1<x<0$	$(-)\ (+)\ (-)$	+ve
$0<x<2$	$(+)\ (+)\ (-)$	-ve
$x>2$	$(+)\ (+)\ (+)$	+ve

We know that a function r(x) is said to be strictly increasing, if f(x) > 0 and it is said to be strictly decreasing, if f(x) < 0. So, the given function f(x) is

i. Strictly increasing

Answer:

Strictly increasing on the intervals (-1, 0), and (2,x)

ii. Strictly decreasing

Answer:

Strictly decreasing on the intervals (-x, -1), and (0,2)

OR

Find the equations of tangent and normal to the curve $x = a\sin^3\theta$, and $y = a\cos^3\theta$ at $\theta = \frac{\pi}{4}$

Answer:

Given, $x = a\sin^3\theta$

On differentiating w.r.t. θ, we get

$$\frac{dx}{d\theta} = a\sin^3\theta \text{, or}$$

$$\frac{dx}{d\theta} = 3a\sin^2\theta\cos\theta$$

Also, $y = a\cos\theta$

By differentiating we get

$$\frac{dy}{d\theta} = 3a\cos^2\theta\sin\theta$$

Now,

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{3a\cos^2\theta\sin\theta}{3a\sin^2\theta\cos\theta} = -\cot\theta$$

At $\theta = \frac{\pi}{4}$

$$\left\{\frac{dy}{dx}\right\}_{\left(\theta=\frac{\pi}{4}\right)} = -1 \quad = -\cot\frac{\pi}{4} \quad = -1$$

Also when, $\theta = \frac{\pi}{4}$, then

$$x = a\left(\sin\frac{\pi}{4}\right)^3, y = a\left(\cos\frac{\pi}{\theta}\right)^3 \text{, or}$$

$$x = a\left(\frac{1}{2}\right)^{\frac{3}{2}}, y = a\left(\frac{1}{2}\right)^{\frac{3}{2}} \qquad \left[\text{when, } \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \text{ and } \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}\right]$$

Now, equation of tangent at the given point is

$$Y - y = \frac{dy}{dx}(X - x) \text{, or}$$

$$Y - \left\{\frac{a}{(2)^{\frac{3}{2}}}\right\} = (-1)\times\left(X - \frac{a}{2^{\frac{3}{2}}}\right) \text{, or}$$

$$Y + X = \frac{2a}{(2)^{\frac{3}{2}}}, \text{ or}$$

$$Y + x = \frac{a}{\sqrt{2}}, \text{ or}$$

$$X + Y - \frac{a}{\sqrt{2}}, \text{ or}$$

$$X + Y - \frac{a}{\sqrt{2}} = 0$$

Also, slope of a normal = $\dfrac{-1}{\text{Slope of tangent}}$

$\therefore$ Equation of normal at given point is $Y - \dfrac{a}{(2)^{\frac{3}{2}}} = (1) \times \left(X - \dfrac{a}{2^{\frac{3}{2}}} \right)$, or

$$X - Y = 0$$

Question: 17

Evaluate: $\int \left(\dfrac{\sin^6 x + \cos^6 x}{\sin^2 x + \cos^2 x} \right) dx$

Answer:

Let

$$L= \int \left(\frac{\sin^6 x + \cos^6 x}{\sin^2 x + \cos^2 x} \right) dx$$

$$= \int \left\{ \frac{\left(\sin^2 x\right)^3 + \left(\cos^2 x\right)^3}{\sin^2 x.\cos^2 x} \right\} dx$$

As because, $a^3 + b^3 = (a+b)^3 - 3ab(a+b)$, then

$$= \int \left[\frac{\left(\sin^2 x + \cos^2 x\right)^3 - \left\{3 \sin^2 x \cos^2 x \left(\sin^2 x + \cos^2 x\right)\right\}}{\sin^2 x \cos^2 x} \right] dx$$

Because, $\sin^2 x + \cos^2 x = 1$, then

$$= \int \left\{ \frac{(1)^3 - 3 \sin^2 x \cos^2 x}{\sin^2 x.\cos^2 x} \right\} dx$$

$$= \int \left(\frac{1}{\sin^2 x.\cos^2 x} \right) dx - 3 \int \left(\frac{\sin^2 x.\cos^2 x}{\sin^2 x.\cos^2 x} \right) dx$$

$$= \int \left\{ \left(\frac{\sin^2 x}{\sin^2 x \cos^2 x} \right) + \left(\frac{\cos^2 x}{\sin^2 x \cos^2 x} \right) \right\} dx - 3 \int (1) dx$$

$$= \int \sec^2 x\, dx + \int \operatorname{cosec}^2 x\, dx - 3 \int 1\, dx$$

$= \tan x - \cot x - 3x + C$

OR

Evaluate: $\int (x-3)\sqrt{x^2+3x-18}\ dx$

Answer:

Let

$i = \int (x-3)\left(\sqrt{x^2+3x-18}\right)dx$

Here we can write (x-3) as

$x - 3 = A\left\{\frac{d}{dx}\left(x^2+3x-18\right)\right\} + B \quad = A(2x+3) + B$

On equating the coefficients of x and constant terms from both sides, we get

2A = 1, and 3A + B = - 3

$\Rightarrow A = \frac{1}{2}$, and $3 \times \frac{1}{2} + B = -3$

$\Rightarrow A = \frac{1}{2}$, and $B = -\frac{3}{2} - 3$

$\Rightarrow A = \frac{1}{2}$, and $B = -\frac{9}{2}$

Thus the given integral reduces in the following form

$L = \int \left\{\frac{1}{2}(2x+3) - \frac{9}{2}\right\} \times \left(\sqrt{x^2+3x+18}\right)dx$

$= \left[\frac{1}{2} \times \int \left\{(2x+3)\sqrt{x^2+3x-18}\right\} dx\right] - \left\{\frac{9}{2} \times \int \left(\sqrt{x^2+3x+18}\right) dx\right\}$

As we assume

$L_1 = \int \left\{(2x+3)\sqrt{x^2+3x-18}\right\}dx$, and

$L_2 = \int \left(\sqrt{x^2+3x+18}\right)dx$

Then we can say,

$L = \left(\frac{1}{2}\right)L_1 - \left(\frac{9}{2}\right)L_2 \qquad \text{(i)}$

Consider,

$L_1 = \int (2x+3)\sqrt{x^2+3x-18}\ dx$

Put $x^2+3x-18 = t$, or

$(2x+3)dx = dt$

$\therefore L_1 = \int t^{\frac{1}{2}}\ dt \quad = \int \frac{2}{3}t^{\frac{3}{2}} + C_1 \quad = \frac{2}{3}\left(x^2+3x-18\right)^{\frac{3}{2}} + C_1$

And

$$L_2 = \int\left(\sqrt{x^2+3x+18}\right)dx$$

$$= \int\left\{\sqrt{\left(x+\frac{3}{2}\right)^2 - 18 - \frac{9}{4}}\right\}dx$$

$$= \int\left\{\sqrt{\left(x+\frac{3}{2}\right)^2 - \frac{81}{4}}\right\}dx$$

$$= \int\left\{\sqrt{\left(x+\frac{3}{2}\right)^2 - \left(\frac{9}{2}\right)^2}\right\}dx$$

$$= \left\{\frac{\left(x+\frac{3}{2}\right)}{2}\times\left(\sqrt{x^2+3x-18}\right)\right\} - \left[\frac{81}{8}\times\left\{\log\left|\left(x+\frac{3}{2}\right)-\sqrt{x^2+3x-18}\right|\right\}\right] + C_2$$

Now, considering $\left(\int\sqrt{x^2-a}\ dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|\right)$ we get

$$= \left\{\left(\frac{2x+3}{4}\right)\times\left(\sqrt{x^2+3x-18}\right)\right\} - \left\{\frac{81}{8}\times\left(\log\left|\frac{2x+3}{2}+\sqrt{x^2+3x-18}\right|\right)\right\} + C_2$$

Plugging the values of L_1,, and L_2 in (i) we get

$$L = \frac{1}{2}\left\{\frac{2}{3}\left(x^2+3x-18\right)^{\frac{3}{2}} + C_1\right\} - \frac{9}{2}\left\{\left(\frac{2x+3}{4}\sqrt{x^2+3x-18}\right) - \left(\frac{81}{8}\log\left|\frac{2x+3}{2}+\sqrt{x^2+3x-18}\right|\right) + C_2\right\}$$

$$= \left\{\frac{1}{3}\times\left(x^3+3x-18\right)^{\frac{3}{2}}\right\} - \left\{\frac{9}{8}\times(2x+3)\times\sqrt{x^2+3x-18}\right\} + \left[\frac{729}{16}\times\left\{\log\left|\frac{2x+3}{2}+\sqrt{x^2+3x-18}\right|\right\}\right] + C$$

Where,

$$C = \left\{\left(\frac{C_1}{2}\right) - \left(\frac{9C_2}{2}\right)\right\}$$

Question: 18

Find the particular solution of the differential equation $\left\{\left(e^x\sqrt{1-y^2}\right)dx + \left(\frac{y}{x}\right)dy\right\} = 0$, given that y=1 when x=0.

Answer:

$e^x\sqrt{1-y^2}\ dx + \frac{y}{x}dy = 0$, or

$$xe^x dx = \left(\frac{-y}{\sqrt{1-y^2}}\right)dy$$

Integrating both sides

$$\int xe^x dx = \frac{1}{2}\int\left(\frac{-2y}{\sqrt{1-y^2}}\right)dy \text{, or}$$

$$xe^x - e^x = \sqrt{1-y^2} + C$$

For x = 0, y = 1, c= -1

$\therefore$ Solution is: $e^x(x-1) = \sqrt{1-y^2} - 1$

Question: 19

Solve the following differential equation: $\left(x^2-1\right)\frac{dy}{dx} + 2xy = \frac{2}{x^2-1}$

Answer:

Given differential equation can be written as: $\frac{dy}{dx} + \left(\frac{2x}{x^2-1}\right)y = \frac{2}{\left(x^2-1\right)^2}$

Integrating factor: $e^{\int \frac{2x}{x^2-1}dx} = e^{\log\left(x^2-1\right)} = x^2-1$

$\therefore$ Solution is

$$y \times \left(x^2-1\right) = \int\left\{\frac{2}{\left(x^2-1\right)^2}\right\} \times \left\{\left(x^2-1\right)dx\right\} + c \;= 2\int\left(\frac{1}{x^2-1}\right)dx + c = \log\left|\frac{x-1}{x+1}\right| + c$$

Question: 20

Prove that, for any three vectors $\vec{a}, \vec{b}, \vec{c}$

$$\left[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}\right] = 2\left[\vec{a}, \vec{b}, \vec{c}\right]$$

Answer:

Starting analysis from L.H.S we get,

$$= \left(\vec{a}+\vec{b}\right) \times \left\{\left(\vec{b}+\vec{c}\right) \times \left(\vec{c}+\vec{a}\right)\right\}$$

$$= \left(\vec{a}+\vec{b}\right)\left\{\left(\vec{b}\times\vec{c}\right) + \left(\vec{b}\times\vec{a}\right) + \left(\vec{c}\times\vec{c}\right) + \left(\vec{c}\times\vec{a}\right)\right\}$$

$$= \vec{a}.\left(\vec{b}\times\vec{c}\right) + \vec{a}.\left(\vec{b}\times\vec{a}\right) + \vec{a}\left(\vec{c}\times\vec{a}\right) + \vec{b}\left(\vec{b}\times\vec{c}\right) + \vec{b}\left(\vec{b}\times\vec{a}\right) + \vec{b}\left(\vec{c}\times\vec{a}\right) \; \left\{\vec{a}\left(\vec{b}\times\vec{a}\right), \vec{a}\left(\vec{c}\times\vec{a}\right)\right\}$$

$$= \vec{b}\left(\vec{b}\times\vec{c}\right) = \vec{b}.\left(\vec{b}\times\vec{a}\right) = 0$$

$$= \left\{\vec{a}.\left(\vec{b}\times\vec{c}\right)\right\} \;= 2\left[\vec{a}, \vec{b}, \vec{c}\right]$$

OR

Vector's $\vec{a}$, $\vec{b}$, and $\vec{c}$ are such that $\vec{a}+\vec{b}+\vec{c} = 0$, and $\left|\vec{a}\right| = 3$, $\left|\vec{b}\right| = 5$, and $\left|\vec{c}\right| = 7$. Find the angle between $\vec{a}$, and $\vec{b}$.

Answer:

Given

$\vec{a}+\vec{b}+\vec{c}=0$

$\therefore \vec{a}+\vec{b}=-\vec{c}$

$\left(\vec{a}+\vec{b}\right)^2=\left(-\vec{c}\right)^2=\left(\vec{c}\right)^2$, or

$\left|\vec{a}\right|^2+\left|\vec{b}\right|^2+2\vec{a}.\vec{b}=\left|\vec{c}\right|^2$, or

$9+25+2\left|\vec{a}\right|\left|\vec{b}\right|\cos\theta=49$, θ being angle between $\vec{a}$, and $\vec{b}$

$\therefore \cos\theta=\dfrac{15}{2.3.5}=\dfrac{1}{2}$, or

$\theta=\dfrac{\pi}{3}$

Question: 21

Show that the lines $\dfrac{x+1}{3}=\dfrac{y+3}{5}=\dfrac{z+5}{7}$, and $\dfrac{x-2}{1}=\dfrac{y-4}{3}=\dfrac{z-6}{5}$ intersect. Also find their point of intersection.

Answer:

Let,

$\dfrac{x+1}{3}=\dfrac{y+3}{5}=\dfrac{z+5}{7}=u$;

$\dfrac{x-2}{1}=\dfrac{y-4}{3}=\dfrac{z-6}{5}=v$

General points on the lines are,
(3u – 1, 5u – 3, 7u – 5), and
(v + 2, 3v + 4, 5v + 6)

Lines intersect (for some u, v) if,
3u – 1 = v + 2,
5u – 3 = 3v + 4,
7u – 5 = 5v + 6

3u – v = 3 (1)
5u – 3v = 7 (2)
7u – 5v = 11 (3)

Solving equations (1), and (2), we get u $=\dfrac{1}{2}$, v = $-\dfrac{3}{2}$

Putting u, and v in equation (3), $7\left(\dfrac{1}{2}\right)-5\left(-\dfrac{3}{2}\right)$ = 11

$\therefore$ Lines intersect

Point of intersection of lines is: $\left(\dfrac{1}{2},-\dfrac{1}{2},-\dfrac{3}{2}\right)$

Question: 22

Assume that each born child is equally likely to be a boy or a girl. If a family has two children, what is the conditional probability that both are girls? Given that

Answer:

Let b_2, g_2 be younger boy and girl
And b_1, g_1 be elder, then sample space of two children is

S = $\{(b_1,b_2),(g_1,g_2),(b_1,g_2),(g_1,b_2)\}$

A (Event that at least one is a girl): $\{(g_1,g_2),(b_1,g_2)\}$

B (Event that at least one is a girl): $\{(g_1,g_2),(b_1,g_2),(g_1,b_2)\}$

E (Event that both are girls): $\{(g_1,g_2)\}$

i. The youngest is a girl.

Answer:

$$P\left(\frac{E}{A}\right)=\frac{P(E\cap A)}{P(A)}=\frac{1}{2}$$

ii. At least one is a girl.

Answer:

$$P\left(\frac{E}{B}\right)=\frac{P(E\cap B)}{P(B)}=\frac{1}{3}$$

***Section C** (Question numbers 23 to 29 carry 6 mark each)*

Question: 23
Two schools P and Q want to award their selected students on the values of discipline, politeness, and punctuality. The school P wants to award ₹x each, ₹y each, and ₹x each, for the respective values to its 3, 2, and 1 students with a total award money of ₹ 1000.

School Q wants to spend ₹1500 to award its 4, 1, and 3 students on the respective values (by giving the same award money for the three values before). If the total amount of awards for one prize on each value is ₹600, using matrices, find the award money for each value. Apart from the above three values, suggest one more value for awards.

Answer:
Here we can say,
$3x + 2y + z = 1000$
$4x + y + 3z = 1500$
$x + y + z = 600$

Therefore,
$$\begin{pmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1000 \\ 1500 \\ 600 \end{pmatrix}, \text{ or}$$
$A . X = B$
$|A| = 3(-2) - 2(1) + 1(3) = -5 \neq 0$
$\therefore X = A^{-1} B$

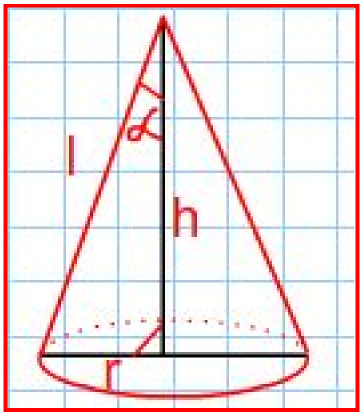
Co – factor's are,
$A_{11} = -2,\quad A_{12} = -1,\quad A_{13} = 3$
$A_{21} = -1,\quad A_{22} = 2,\quad A_{23} = -1$
$A_{31} = 5,\quad A_{32} = -5,\quad A_{33} = -5$

Hence,
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{5}\begin{pmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{pmatrix}\begin{pmatrix} 1000 \\ 1500 \\ 600 \end{pmatrix}$$
$\therefore x = 100,\ y = 200,\ z = 300$

i.e., ₹100 for discipline, ₹ 200 for politeness, and ₹ 300 for punctuality. One more value like sincerity, truthfulness, etc.

Question: 24
Show that the semi-vertical angle of the cone of the maximum volume, and of given slant height is $\cos^{-1}\dfrac{1}{\sqrt{3}}$.

Answer:



For correct figure, let consider radius, height and slant height of cone as r, h, and l. Then, $r^2 + h^2 = l^2$, where l is constant

Volume of cone (V): $\frac{1}{3}\pi r^2 h$

$\therefore\ V = \frac{\pi}{3}h\left(l^2 - h^2\right) = \frac{\pi}{3}\left(l^2 h - h^3\right)$

$\frac{dv}{dh} = \frac{\pi}{3}\left(l^2 - 3h^2\right)$

$\therefore \frac{dv}{dh} = 0 \Rightarrow h = \frac{l}{\sqrt{3}}$

$\frac{d^2v}{dh^2} = -2\pi h = -2\pi \times \left(\frac{l}{\sqrt{3}}\right) = -\frac{2\pi l}{\sqrt{3}} < 0$

$\therefore$ at $h = \frac{l}{\sqrt{3}}$, volume is maximum

$\cos\alpha = \frac{h}{l} = \frac{1}{\sqrt{3}}$

$\therefore \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

Question: 25

Evaluate $\int_{\pi/6}^{\pi/3}\left(\frac{dx}{1+\sqrt{\cot x}}\right)$

Answer:

Let,

$$L = \int_{\pi/6}^{\pi/3}\left(\frac{dx}{1+\sqrt{\cot x}}\right) = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\left(\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\right)dx \qquad (1)$$

We know that $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

On applying this property in (1) we get

$$L = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left\{ \frac{\sqrt{\sin\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)}+\sqrt{\cos\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)}} \right\} dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}} dx$$

As $\cos\left(\frac{\pi}{\theta}-x\right) = \sin x$, and $\sin\left(\frac{\pi}{\theta}-x\right) = \cos x$, therefore we can simplify the equation as,

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}} \right) dx \qquad (2)$$

Now adding (1), and (2) we get

$$2L = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}} dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (1)\, dx = (x)_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{\pi}{3}-\frac{\pi}{6} = \frac{\pi}{6}, \text{ or}$$

$$L = \frac{\pi}{12}$$

Question: 26

Find the area of the region in the first quadrant enclosed by the x-axis the line y = x and the circle $x^2 + y^2 = 32$

Answer:

The given quadrants of curves are

$y = x$ (i)

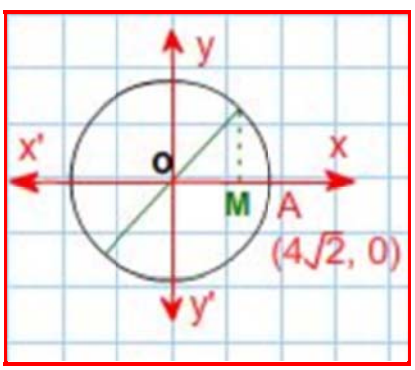
and

$x^2 + y^2 = 32$ (ii)

On solving (i), and (ii), we get

x = 4, and

y = 4



Required area:

Area of the region OABO = Area of the region OBMO + Area of the region BMAB (iii)

Now,

Area of the region OBMO: $\int_0^4 y\,dx = \int_0^4 x\,dx = \frac{1}{2}\left[x^2\right]_0^4 = 8$ (iv)

And area of the region BMAB

$= \int_4^{4\sqrt{2}} y\,dx$

$= \int_4^{4\sqrt{2}} \sqrt{32-x^2}\,dx$

$= \int_4^{4\sqrt{2}} \sqrt{\left(4\sqrt{2}\right)^2 - x^2}\,dx$

$= \left\{\left(\frac{1}{2}x\sqrt{32-x^2}\right)+\left(\frac{1}{2}\times 32\times \sin^{-1}\frac{x}{4\sqrt{2}}\right)\right\}_4^{4\sqrt{2}}$ $\left[\because \int\sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$

$= \left[\frac{1}{2}\times 4\sqrt{2}\times 0+\frac{1}{2}\times 32\times \sin^{-1}(1)\right]-\left[\frac{4}{2}\sqrt{32-16}+\frac{1}{2}\times 32\times \sin^{-1}\frac{1}{\sqrt{2}}\right]$

$= \left(16\times\frac{\pi}{2}\right)-\left(2\times 4+16\times\frac{\pi}{4}\right)$

$= 8\pi-(8+4\pi)$

$= 4\pi - 8$ (v)

On putting the values from (iv), and (v) in (iii), we get

Required area: 4π

Question: 27

Find the distance between the point (7, 2, 4) and the plane determined by the points A(2, 5, -3), B(-2, -3, 5) and C(5, 3, -3).

Answer:

Equation of plane through points A, B, and C is

$\begin{vmatrix} x-2 & y-5 & z+3 \\ -4 & -8 & 8 \\ 3 & -2 & 0 \end{vmatrix} = 0$, or

$16x+24y+32z-56=0$, or

i2x + 3y + 4z – 7 = 0

Distance of plane from (7,2,4): $\left|\frac{2(7)+3(2)+4(4)-7}{\sqrt{9+16+4}}\right| = \sqrt{29}$

OR

Find the distance of the point (-1, -5, -10) from the point of intersection of the line $\vec{r} = 2\hat{i}-\hat{j}+2\hat{k}+\lambda\left(3\hat{i}+4\hat{j}+2\hat{k}\right)$, and the plane $\vec{r}.\left(\hat{i}-\hat{j}-\hat{k}\right)=5$

Answer:

General point of the line is $(2+3\lambda)\hat{i}+(-1+4\lambda)\hat{j}+(2+2\lambda)\hat{k}$

Putting in the equation of plane; we get $1.(2+3\lambda)-1.(-1+4\lambda)+1.(2+2\lambda)=5$

$\therefore \lambda = 0$

Point of intersection: $2\hat{i}-\hat{j}+2\hat{k}$ or (2, -1, 2)

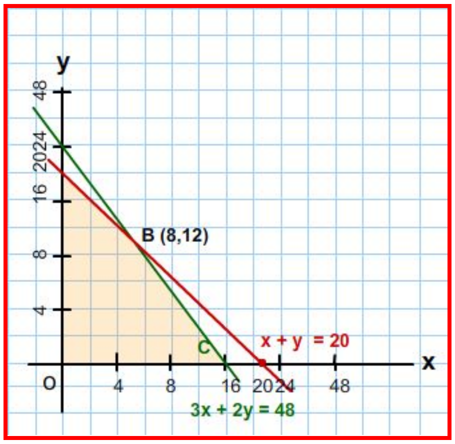
Distance: $\sqrt{(2+1)^2+(-1+5)^2+(2+10)^2} = \sqrt{169}$ = 13

Question: 28

A dealer in rural area wishes to purchase a number of sewing machines. He has only ₹5760 to invest and has space for at most 20 items for storage. An electronic sewing machine cost him ₹360 and a manually operated sewing machine ₹240.

He can sell an electronic sewing machine at a profit of ₹22, and manually operated sewing machine at a profit `18. Assuming that he can sell all the items that he can buy, how should he invest his money in order to maximize his profit? Make it as LPP, and solve graphically.

Answer:



Let x, and y be electronic and manually operated sewing machines purchased respectively

$\therefore$ L.P.P is Maximize P = 22x + 18y

Subject to 360x + 240y $\leq$ 5760, or

3x + 2y $\leq$ 20

x $\geq$ 0, y $\geq$ 0

For correct graph of feasible region are

A (0, 20),

B(8, 12),

C(16, 0), and

O(0, 0)

P(A) = 360,

P(B) = 392,

P(C) = 352

$\therefore$ For Maximum P, Electronic machines: 8

Manual machines: 12

Question: 29

A card from a pack of 52 playing cards is lost. From the remaining cards of the pack three cards are drawn at random (without replacement), and are found to be all spades. Find the probability of the lost card being a spade.

Answer:

Let E_1 : Event that lost card is a spade

E_2 : Event that three spades are drawn without replacement from 51 cards.

$$P(E_1) = \frac{13}{52} = \frac{1}{4}$$

$$P(E_2) = 1 - \frac{1}{4} = \frac{3}{4}$$

$$P\left(\frac{A}{E_1}\right) = \frac{{}^{12}C_3}{{}^{51}C_3}$$

$$P\left(\frac{A}{E_2}\right) = \frac{{}^{13}C_3}{{}^{51}C_3}$$

$$P\left(\frac{E_1}{A}\right) = \frac{\left\{\frac{1}{4} \times \left(\frac{{}^{13}C_3}{{}^{51}C_3}\right)\right\}}{\left\{\frac{1}{4} \times \left(\frac{{}^{13}C_3}{{}^{51}C_3}\right)\right\} + \left\{\frac{3}{4} \times \left(\frac{{}^{13}C_3}{{}^{51}C_3}\right)\right\}} = \frac{10}{49}$$

OR

From a lot of 15 bulbs which include 5 defectives, a sample of 4 bulbs is drawn by one with replacement. Find the probability distribution of number of defective bulbs. Hence find the mean of the distribution.

Answer:

X (No. of defective bulbs out of 4 drawn): 0,1,2,3,4

Probability of defective bulb: $\frac{5}{15} = \frac{1}{3}$

Probability of a non-defective bulb: $1 - \frac{1}{3} = \frac{2}{3}$

Probability distribution is:

X	0	1	2	3	4
P(x)	$\frac{16}{81}$	$\frac{32}{81}$	$\frac{24}{81}$	$\frac{8}{81}$	$\frac{1}{81}$
X P(x)	0	$\frac{32}{81}$	$\frac{48}{81}$	$\frac{24}{81}$	$\frac{4}{81}$

Mean $\{\sum XP(x)\} : \frac{108}{81} = \frac{4}{3}$