2013

Section A (Compulsory) *(Question numbers 1 to 7)*

Question: 1 [3 x 10 = 30]

i. If (A-2I)(A-3I) = 0, where A = $\begin{pmatrix} 4 & 2 \\ -1 & x \end{pmatrix}$ and I = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, find the value of x.

Answer:

Given $A = \begin{bmatrix} 4 & 2 \\ -1 & x \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(A-2I) (A-3I) = 0

$$\left\{\begin{bmatrix} 4 & 2 \\ -1 & x \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right\}\left\{\begin{bmatrix} 4 & 2 \\ -1 & x \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right\} = 0$$

$$\begin{bmatrix} 4 & -2\ 2 & 0 \\ -1 & -2\ x & 2 \end{bmatrix}\begin{bmatrix} 4 & -3\ 2 & 0 \\ -1 & -2\ x & 2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 4 & 2 \\ -1 & x-2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & x-3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 2 & -2 & 4+2x+6 \\ -1 & -x+2 & -2+x^2-5x+6 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & 2x-2 \\ -x+1 & x2-5x+4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

i.e. $-x+1 = 0$

x = 1.

ii. Find the value(s) of κ so that the line $2x + y + \kappa = 0$ may touch the hyperbola $3x^2 - y^2 = 3$

Answer:

Equation of hyperbola,

$3x^2 - y^2 = 3$

$$\frac{3x^2}{3} - \frac{y^2}{3} = 1$$

$$\frac{x^2}{1} - \frac{y^2}{3} = 1$$

$$\frac{x^2}{1^2} - \frac{y^2}{3} = 1$$

$$\frac{x^2}{(1)^2} - \frac{y^2}{\left(\sqrt{3}\right)^2} = 1$$

Compare with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get

a = 1 and b = $\sqrt{3}$

Equation of line, 2x + y + k = 0 (given)

y = -2x – k

Compare with y = mx + c, we get

m = -2 and c = -k

If given equation touches hyperbola, then

$c = \pm\sqrt{a^2m^2 - b^2}$

$-k = \pm\sqrt{(1)^2.(-2)^2 - (\sqrt{3})^2}$

$-k = \pm\sqrt{4-3}$

-k = $\pm$ 1
k = 1

iii. Prove that: $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \frac{1}{2}\sin^{-1}\frac{4}{5}$

Answer:

L.H.S = $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9}$

We know that

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy)}\right)$$

$$= \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \tan^{-1}\left(\frac{\frac{1}{4}+\frac{2}{9}}{1-\frac{1}{4}x\frac{2}{9}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{17}{36}}{1-\frac{2}{36}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{17}{36}}{\frac{34}{36}}\right)$$

$$= \tan^{-1}\frac{17}{34}$$

$\therefore$ Let $\tan^{-1}\frac{1}{2}$ = A then

tan A = $\frac{1}{2}$

$\therefore \sin A = \frac{1}{\sqrt{5}}$ and $\cos A = \frac{2}{\sqrt{5}}$

We know that,
sin 2A = 2 sin A cos A

$= 2x\frac{1}{\sqrt{5}}x\frac{2}{\sqrt{5}}$

$\sin 2A = \frac{4}{5}$

$2A = \sin^{-1}\frac{4}{5}$

$A = \frac{1}{2}\sin^{-1}\frac{4}{5}$

Hence, L.H.S = R.H.S

iv. Using L'Hospital's Rule, evaluate:

$$\lim_{x \to 0}\left(\frac{e^x - e^{-x} - 2x}{x - \sin x}\right)$$

Answer:

Given: $\lim_{x \to 0}\left(\frac{e^x - e^{-x} - 2x}{x - \sin x}\right)$

This is also $\frac{0}{0}$ form as $e^0 - e^{-0} - 2 \times 0 = 1-1 = 0$

Differentiating, $\lim_{x \to 0}\left(\frac{e^x - e^{-x} - 2}{x - \sin x}\right)$ $\left[\text{form } \frac{0}{0}\right]$

again differentiating, $\lim_{x \to 0}\left(\frac{e^x - e^{-x}}{x - \sin x}\right)\left[\text{form } \frac{0}{0}\right]$

$\lim_{x \to 0}\left(\frac{e^x - e^{-x}}{x - \sin x}\right)$

$= \frac{1+1}{\cos} = \frac{2}{1} = 2$

v. Evaluate : $\int \frac{1}{x + \sqrt{x}} dx$

Answer:

Given $\int \frac{1}{x + \sqrt{x}} dx$

Putting $\sqrt{x}$ = t or x = t2

$\therefore \int \frac{2tdt}{t^2 + t} = 2\int \frac{2tdt}{t+1}$

= 2 log (t+1) + c

= 2 log $\left(\sqrt{x} + 1\right) + c$

vi. Evaluate : $\int_0^1 \log\left(\frac{1}{x} - 1\right) dx$

Answer:

Given $\int_0^1 \log\left(\frac{1}{x} - 1\right) dx$

Let $I = \int_0^1 \log\left(\frac{1-x}{dx}\right)$

By using $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$I = \int_0^1 \log\left\{\frac{1-(1-x)}{1-x}\right\} dx$

$I = \int_0^1 \log\left\{\frac{x}{1-x}\right\}$

Adding (i) and (ii), we get

$$2I=\int_0^1\left[\log\left(\frac{1-x}{x}\right)+\log\left(\frac{x}{1-x}\right)\right]dx$$

$$=\int_0^1\log\left(\frac{1-x}{x}x\frac{x}{1-x}\right)dx$$

$= \int_0^1 \log 1dx$

$= 0^0$.

vii. Two regression lines are represented by 4x + 10y = 9 and 6x + 3y = 4. Find the line of regression of y on x.

Answer:

Let regression lines are

4x + 10y = 9

6x + 3y = 4

And solving equation (i) and (ii) we get

$$x=\frac{13}{48}\ y=\frac{19}{24}$$

$$\bar{x}==\frac{13}{48}\ \bar{y}=\frac{19}{24}$$

$$y=-\frac{4}{10}x+\frac{9}{10}$$

$$b_{yx}\ =\frac{4}{10}$$

Now equation (i),

$$y=-\frac{4}{10}x+\frac{9}{10}$$

$$\therefore\ b_{yx}=-\frac{4}{10}$$

Line of Regression of y and x,

$$y-\bar{y}=b_{yx}\left(x-\bar{x}\right)$$

$$y-\frac{19}{24}=-\frac{4}{10}\left(x-\frac{13}{48}\right)$$

$$10y-\frac{190}{24}=-4x+\frac{52}{48}$$

$$4x+10y=\frac{190}{24}+\frac{52}{48}$$

4x + 10y = 0.

viii. If 1, ω and ω^2 are the cube roots of unity, evaluate $(a - w^4 + w^8)(1-w^8 + w^{16})$

Answer:

$(1-\omega^4 + \omega^8)(1-\omega^8 + \omega^{16})$

$= [1-\omega^3.\omega + (-\omega^3)^2. -\omega^2] [1-(-\omega^3)^2. \omega^2 + (-\omega^3)^5. -\omega^2]$

$=[1-\omega + (1)^2. \omega^2] [1-(1)^2.\omega^2 + (1)^5. \omega]$

$= (1- \omega^2 - \omega)(1+ \omega -\omega^2)$

$=(-\omega-\omega)(-\omega^2 - \omega^2)$

$= 4\omega^3$

$= 4(1)^3 = 4$

ix. Solve the differential equation:

$$\log\left(\frac{dy}{dx}\right) = 2x - 3y$$

Answer:

Given: $\log\left(\frac{dy}{dx}\right) = 2x - 3y$

$$\frac{dy}{dx} = e^{2x-3y} = \frac{e^{2x}}{e^{3y}}$$

$e^{3y}\,dy = e^{2x}\,dx$

Now integrating both sides, we get

$$\int e^{3y}\,dy = \int e^{2x}\,dx$$

$$\frac{e^{3y}}{3} = \frac{e^{2x}}{2} + c$$

x. If two balls are drawn from a bag containing three red balls and four blue balls, find the probability that:

a. They are of the same colour.
b. They are of different colours.

Answer:

Number of red balls = 3
Number of blue balls = 4
Total number of balls = 3 + 4 = 7

a. P(Balls are of same colour) = P (Both Red Ball) + P(Both Blue Ball)

$$\frac{{}^3C_2}{{}^7C_2} + \frac{{}^4C_2}{{}^7C_2} = \frac{3.2}{7.6} + \frac{4.3}{7.6} = \frac{18}{42} = \frac{3}{7}$$

b. P(Balls are of different colour) = P (One Red Ball and One Blue Ball)

$$\frac{{}^3C_1 \times {}^4C_1}{{}^7C_2} = \frac{3 \times 2 \times 2}{7 \times 6} = \frac{4}{7}$$

Question: 2 [5+5=10]

a. Using properties of determinants, prove that:

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ y+z & z+x & x+y \end{vmatrix} = (x-y)(y-z)(z-x)(x+y+z)$$

Answer:

$$\text{L.H.S} = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ y+z & z+x & x+y \end{vmatrix} = (x-y)(y-z)(z-x)(x+y+z)$$

Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$

$$= \begin{vmatrix} x & y-x & z-x \\ x^2 & y^2-x^2 & z^2-x^2 \\ y+z & -(y-x) & x-z \end{vmatrix}$$

Taking (y-x) from C_2 and (z-x) from C_3 as common

$$=(y-x)(z-x)\begin{vmatrix} x & 1 & 1 \\ x^2 & y+x & z+x \\ y+z & -1 & -1 \end{vmatrix}$$

Applying $R_3 \to R_3 - R_1$

$$=(y-x)(z-x)\begin{vmatrix} x & 1 & 1 \\ x^2 & y+x & z+x \\ x+y+z & 0 & 0 \end{vmatrix}$$

Taking (x+y+z) as common from R3,

$$=(y-x)(z-x)(x+y+z)\begin{vmatrix} x & 1 & 1 \\ x^2 & y+x & z+x \\ 1 & 0 & 0 \end{vmatrix}$$

= (y-x) (z-x) (x+y+z) [1.(z+x) – 1 (y+x)]
= (y-z) (z-x) (x+y+z) [z+x-y-x]
= (y-z)(z-x) (x+y+z) (z-y)
=(x-y) (y-z) (z-x) (x+y+z)
= R.H.S.

b. Find A-1, where A = $\begin{bmatrix} 4 & 2 & 3 \\ 1 & 1 & 1 \\ 3 & 1 & -2 \end{bmatrix}$

4x+2y+3z =2
x+y+z =1
3x+y+2z = 5

Answer:
Given:

$$A = \begin{bmatrix} 4 & 2 & 3 \\ 1 & 1 & 1 \\ 3 & 1 & -2 \end{bmatrix}$$

$|A|$ = 4(-2-1)-2(-2-3) +3(1-3)
=4(-3)-2(-5)+3(-2)
=-12+10-6
=-8

$$A_{11} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} = -2-1=-3$$

$$A_{12} = \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix} = -(-2-3)=-3$$

$$A_{13} = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} = 1-3=-2$$

$$A_{21}=\begin{bmatrix}2 & 3\\ 1 & -2\end{bmatrix}=-(-4-3)=7$$

$$A_{22}=\begin{bmatrix}4 & 3\\ 3 & -2\end{bmatrix}=-8-9=-17$$

$$A_{23}=\begin{bmatrix}4 & 2\\ 3 & 1\end{bmatrix}=-(4-6)=-2$$

$$A_{31}=\begin{bmatrix}2 & 3\\ 1 & 1\end{bmatrix}=2-3=-1$$

$$A_{32}=\begin{bmatrix}4 & 3\\ 1 & 1\end{bmatrix}=-(4-3)=-1$$

$$A_{33}=\begin{bmatrix}4 & 2\\ 3 & 1\end{bmatrix}=4-2=2$$

$$A-1=\frac{adj\,A}{|A|}$$

$$=\frac{1}{-8}\begin{bmatrix}-3 & 7 & -1\\ 5 & -17 & -1\\ -2 & 2 & 2\end{bmatrix}\begin{bmatrix}2\\ 1\\ 5\end{bmatrix}$$

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix}=-\frac{1}{8}\begin{bmatrix}-6 & +7 & -5\\ 10 & -17 & -5\\ -4 & +2 & +10\end{bmatrix}$$

$$=\begin{bmatrix}-4\\ -12\\ 8\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\\ \frac{3}{2}\\ -1\end{bmatrix}$$

$$x=\frac{1}{2}$$

$$y=\frac{3}{2}$$

Question: 3 [5+5=10]

a. Solve for x: $\sin^{-1}x + \sin^{-1}(1-x) = \cos^{-1}x$

Answer:

Given:

$\sin^{-1}x + \sin\text{-}1(1-x) = \cos^{-1} x$

$$\Rightarrow \sin^{-1}[x\sqrt{1-(1-x)^2}+(1-x)\sqrt{1-x^2} = \cos^{-1} x$$

$$=\sin^{-1}\left[\sqrt{2x-x^2}+(1-x)\sqrt{1-x^2}\right]=\sin^{-1}\sqrt{1-x^2}$$

$$\Rightarrow x\sqrt{2x-x^2}+(1-x)\sqrt{1-x^2}=\sqrt{1-x^2}$$

$$\Rightarrow x\sqrt{2x-x^2}=\sqrt{1-x^2}-(1-x)\sqrt{1-x^2}$$

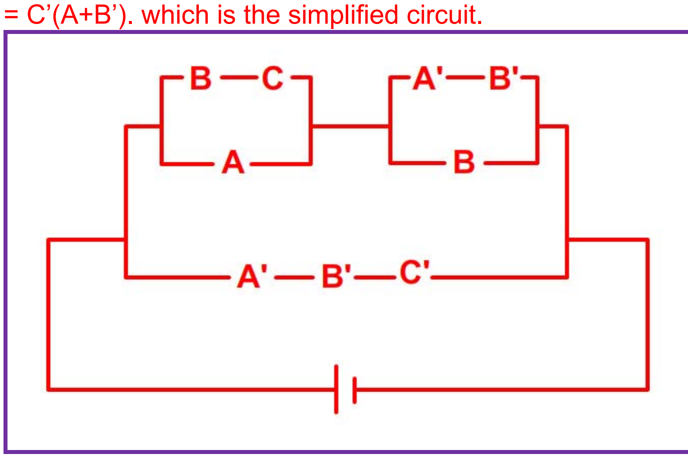
$\Rightarrow x\sqrt{2x-x^2} = \sqrt{1-x^2}\,[1-(1-x)]$
$\Rightarrow x\sqrt{2x-x^2} = \sqrt{1-x^2}$
$\Rightarrow x\sqrt{2x-x^2} = \sqrt{1-x^2} = 0$
$\Rightarrow x\left[\sqrt{2x-x^2} = \sqrt{1-x^2}\right] = 0$
$\Rightarrow x = 0$
or $\sqrt{2x-x^2} = \sqrt{1-x^2}$
$\sqrt{2x-x^2} = \sqrt{1-x^2}$
Taking square on both sides, we get
2x – x2 = 1 – x2
$2x = 1 \Rightarrow x = \frac{1}{2}$
Hence, $x = 0, \frac{1}{2}$

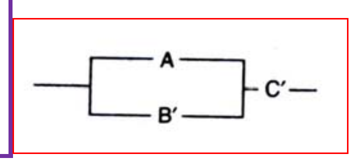
b. Construct a circuit diagram for the following Boolean Function:

(BC+A)(A’+B’+C’) + A’B’C’

Using laws of Boolean Algebra, simplify the function and draw the simplified circuit.

Answer:
(BC+A)(A’+B’+C’) + A’B’C’
= BC(A’B’ + C’) + A(A’B’) + C’) + A’B’C’
= BC A’B’ + BCC’ + AA’B’ + AC’ + A’B’C’
= BB’ A’C + B X 0 + 0 X B’ + AC’ + A’B’C’
0 X A’C + 0 + 0 + AC’ + A’B’C’
= C’ (A + A’B’)
= C’ [(A + A’) . (A + B’)]
= C’ [1.(A+B’)]
= C’(A+B’). which is the simplified circuit.





Question: 4 [5+5=10]

a. Verify Lagrange’s Mean Value Theorem for the function $f(x) = \sqrt{x^2 - x}$ in the interval [1,4].

Answer:

Given f(x) = $\sqrt{x^2 - x}$

$$f'(x) = \frac{2x-1}{2\sqrt{x^2-x}}$$

We know that a polymonial function is continuous and differentiable. So, f(x) being a polynomial is continuous on [1,4] and differentiable on (1,4). Thus, f(x) satisfies both the conditions of Lagrange's mean value theorem on [1,4].

So , there must exist at least one real number c ε (1,4).

Such that $f'(c) = \frac{f(4)-f(1)}{4-1}$

$$\frac{2c-1}{2\sqrt{c^2-c}} = \frac{2\sqrt{3}-0}{3} = \frac{2\sqrt{3}}{3}$$

$$6c - 3 = \left(4\sqrt{3}\sqrt{c^2-c}\right)$$

Taking square on both sides, we get

$$(6c-3)^2 = \left(4\sqrt{3}\sqrt{c^2-c}\right)$$

$36c^2 + 9 – 36c = 48 (c^2\text{-}c)$

$36c^2 + 9 – 36c = 48c^2 - 48c$

$36c^2 – 48c^2 + 9 – 36c + 48c = 0$

$-12c^2 + 12c + 9 = 0$

$4c^2 – 4c – 3 = 0$

2c(2c-3) + 1 (2c-3) = 0

2c = 3 or 2c = -1

C = $\frac{3}{2}$ = 1.5.

Thus c ε (1,4).

Hence, Lagrange's mean value theorem is satisfied.

b. Find the following information, find the equation of the Hyperbola and the equation of its Transverse Axis:

Focus: (-2,1), Directrix: 2x-3y+1 = 0, e = $\frac{2}{\sqrt{3}}$

Answer:

Given: Focus (-2,1).

Directrix, 2x- 3y = +1 = 0 and eccentricity, e = $\frac{2}{\sqrt{3}}$

$\therefore$ according to definition, let P(x,y) be any point in the conic.

Distance of focus from P = (Distance of P from directrix) x e

$$\sqrt{(x+2)^2 + y(y-1)^2} = \frac{2}{\sqrt{3}}\left(\frac{2x-3y+1}{\sqrt{4+9}}\right)$$

$$\sqrt{(x+2)^2 + y(y-1)^2} = \frac{2(2x-3y+1)}{\sqrt{3}.\sqrt{13}}$$

Taking square on both sides, we get

$39(x^2 + 4 + 4x + y^2 + 1 – 2y) = 4(4x^2 + 9y^2 + 1 – 12xy – 6y – 4x)$

$39x^2 + 195 + 156x + 39y^2 – 78y = 16x^2 + 36y^2 + 4 – 48xy – 24y – 16x$

$\Rightarrow 39x^2 – 16x^2 + 39y^2 + 156x + 16x – 78y + 24y + 195 – 4 + 48xy = 0$

$\Rightarrow 23x^2 + 3y^2 + 172x - 54y + 48xy + 191 = 0.$
Transverse axis perpendicular to directrix.
Equation of Transverse axis.
$3x + 2y + k = 0$
It passes through (-2,1)
$3 \times (-2) + 2 \times 1 + k = 0$
$-6 + 2 + k = 0$
$K = 4$
$\therefore$ equation of transverse axis is
$3x + 2y + 4 = 0.$

Question: 5 [5+5=10]

a. If $y = \left(\cot^{-1} x\right)^2$, show that $\left(1+x^2\right)^2 \frac{d^2y}{dx^2} + 2x\left(1+x^2\right)\frac{dy}{dx} = 2$

Answer:

Given: $y = \left(\cot^{-1} x\right)^2$

Differentiating, $\frac{dy}{dx} = 2\cot^{-1} x \times \left(-\frac{1}{1+x^2}\right)$

Again, differentiating, $\left(1+x^2\right)\frac{dy}{dx} = -2\cot^{-1} x$

$\left(1+x^2\right)\frac{d^2y}{dx^2} + \frac{dy}{dx} \times 2x = 2 \times \frac{1}{1+x^2}$

$\left(1+x^2\right)^2 \frac{d^2y}{dx^2} + 2x\left(1+x^2\right)\frac{dy}{dx} = 2$

b. Find the maximum volume of the cylinder which can be inscribed in a sphere of radius $3\sqrt{3}$. (Leave the answer in terms of π)

Answer:
Let radius of a cylinder ABCD be r and height be h, and 2R be diameter of inscribe circle, then
$\because$ From ΔABCD, $(2r)^2 + h^2 = (2R)^2$
$4r^2 = 4R^2 - h^2$

$r^2 = R^2 - \frac{h^2}{4}$

maximum volume of the cylinder

$V = \pi r^2 h$

$= \pi\left(R^2 - \frac{h^2}{4}\right)$

$= \pi\left(R^2 h - \frac{h^3}{4}\right)$ ....(i)

$\therefore \frac{dV}{dh} = \pi\left(R^2 - \frac{3h^2}{4}\right)$

From maxima and minima, we get

$$\frac{dV}{dh} = 0$$

$$R^2 - \frac{3h^2}{4} = 0$$

$$\frac{3h^2}{4} = R^2 \Rightarrow h^2 = \frac{4R^2}{3}$$

$$h = \frac{2R}{\sqrt{3}}$$

From equation (i)

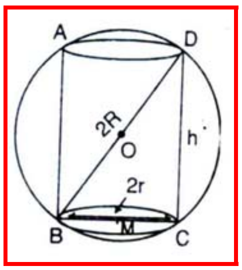
$$V = \pi\left(R^2h - \frac{h^3}{4}\right)$$

$$= \pi\left(\frac{2R^3}{\sqrt{3}} - \frac{8R^3}{4 \times 3\sqrt{3}}\right)$$

$$= \frac{4\pi R^3}{3\sqrt{3}}$$

$$\frac{4\pi \times 3\sqrt{3} \times 3\sqrt{3} \times 3\sqrt{3}}{3\sqrt{3}} \qquad \text{Given} : R = 3\sqrt{3}$$

= 208π cu cm.



Question: 6 [5+5=10]

a. Evaluate: $\int \frac{\cos^{-1} x}{x^2} dx$

Answer:

Given: $\int \frac{\cos^{-1} x}{x^2} dx$

Put cos-1x = 0, then

X = cos θ

dx = -sinθ dθ

$$I = -\int \frac{\theta}{\cos^2\theta} . \sin\theta \, d\theta$$

$$= -\int \theta.(\sec\theta.\tan\theta)d\theta$$

$$= -\theta(\sec\theta) - \int 1.(\sec\theta)d\theta$$

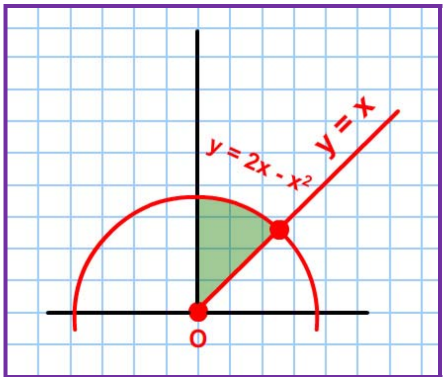
= -θ (sec θ) - $\int$1.(sec θ. tan θ)dθ (on integrating by parts)

$$= -\theta\sec\theta + \log\left|\sec\theta + \tan\theta\right| + c$$

$$= -\frac{\cos^{-1} x}{x} + \log\left|\frac{1}{x} + \frac{\sqrt{1-x^2}}{x}\right| + c$$

b. Find the area bounded by the curve y = 2x – x2 and the line y= x.

Answer:



Equation of curve $y = 2x - x^2$ ….(i)

And equation of line, $y = x$ ….(ii)

Putting y = x in equation (i), we get

$X = 2x - x^2$

$-x = -x^2$

$x^2 - x = 01$

$x(x-1) = 0$

$x = 0, 1.$

Required area = $\int_0^1 (2x - x^2 - x)dx$

$= \int_0^1 (x - x^2)dx$

$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]$

$= \frac{1}{6}$ sq. unit.

Question: 7 [5+5=10]

a. Find the Karl Perarson's Coefficient of Correlation between x and y for the following data:

x	16	18	21	20	22	26	27	15
y	22	25	24	26	25	30	33	14

Answer:

x	y	u = x – 20	v = y -25	u2	v2	uv
16	22	-4	-3	16	9	12
18	25	-2	0	4	0	0
21	24	-1	-1	1	1	1
20	26	0	1	0	1	0
22	25	2	0	4	0	0
26	30	6	5	36	25	30
27	33	7	8	49	64	56
15	14	-5	-11	25	121	55
165	199	3	-1	135	221	154

Here, $\bar{x} = \frac{165}{8} = 20.6$

and $\bar{y} = \frac{199}{8} = 24.8$

$$r = \frac{\sum uv - \frac{\sum u \sum v}{n}}{\sqrt{\left\{\sum u^2 - \frac{(\sum u^2)}{n}\right\}\left\{\sum v^2 - \frac{(\sum v^2)}{n}\right\}}}$$

$$= \frac{154 - \frac{3x - 1}{8}}{\sqrt{\left\{135 - \frac{9}{8}\right\}\left\{221 - \frac{1}{8}\right\}}}$$

$$= \frac{\frac{1232 + 3}{8}}{\sqrt{\left\{\frac{1080 - 9}{8}\right\}\left\{\frac{1768 - 1}{8}\right\}}}$$

$$= \frac{\frac{1235}{8}}{\sqrt{\frac{1071}{8} \times \frac{1767}{8}}}$$

$$= \frac{1235}{32.73 \times 42.04}$$

$$= \frac{1235}{1375} = 0.898$$

b. The following table shows the mean and standard deviation of the marks of Mathematics and Physics scored by the students in a school:

	Mathematics	Physics
Mean	84	81
Standard Deviation	7	4

The correlation co-efficient between the given marks is 0.86. estimate the likely marks in Physics if the marks in Mathematics are 92.

Answer:

Given : $\bar{x} = 84$, $\bar{y} = 81$, $\sigma_y = 4$, and $r = 0.86$

$$b_{yx} = r.\frac{\sigma_y}{\sigma_x} = \frac{0.86 \times 4}{7} = 0.49$$

$$b_{xy} = r.\frac{\sigma_x}{\sigma_y} = \frac{0.86 \times 7}{4} = 1.505$$

Regression equation of y on x is

$y - \bar{y} = b_{yx}(x - \bar{x})$

OR $y - 81 = 0.49 (x - 84)$
OR $y - 81 = 0.49x - 41.16$
OR $y = 0.49 x - 39.84$
Now putting x = 92, then we get
$y = 0.49 \times 92 \times 39.84$
$y = 45.08 + 39.84$
$y = 84.92$
Hence likely marks in physics in 84.92.

Question: 8 [5+5=10]

a. Bag A contains three red and four white balls; bag B contains two red and three white balls. If one ball is drawn from bag A and two balls from bag B, find the probability that:

i. One ball is red and two balls are white;

Answer:
Two white and one red ball can be drawn in two mutually exclusive ways :
Drawing one white ball from bag A and two balls from bag B out of which one is white and other is red.
Drawing one red ball from bag A and two white balls from B.

Let E_1 = Drawing a white ball from bag A
E_2 = Drawing one red and one white ball from bag B.
E_3 = Drawing one red ball from bag A.
E_4 = Drawing two white balls from B.

Then $P(E_1) = \frac{4}{7}$, $P(E_2) = \frac{C_1^2 \times C_1^3}{C_2^5} = \frac{3}{5}$

$P(E_3) = \frac{3}{7}$, $P(E_4) = \frac{C_2^3}{C_2^5} = \frac{3}{10}$

Now P (Two white balls and one red ball)

$= P(E_1).P(E_2) + P(E_3).P(E_4)$ (by using multiplication theorem)

$$= \frac{4}{7} \times \frac{3}{5} + \frac{3}{7} \times \frac{3}{10}$$

$$= \frac{12}{35} + \frac{9}{70} = \frac{33}{70}$$

ii. All the three balls are of the same colour.

Answer:
= P (All White Balls) + P (All Blue Balls)

$$= \frac{4}{7} \times \frac{C_2^3}{C_2^5} + \frac{3}{7} \times \frac{C_2^2}{C_2^5}$$

$$= \frac{4}{7} \times \frac{3}{10} + \frac{3}{7} \times \frac{1}{10}$$

$$= \frac{12}{70} + \frac{3}{70} = \frac{15}{70} = \frac{3}{14}$$

b. Three persons Aman, Bipin and Mohan attempt a Mathematics problem independently. The odds in favour of Aman and Mohan solving the problem are 3:2 and 4:1 respectively and the odds against Bipin solving the problem are 2:1. Find:

i. The probability that all the three will solve the problem.

ii. The probability that problem will be solved.

Answer:

Odds in favour of Aman ; 3:2.

P_1 = Prob. That Aman will solve = $\frac{3}{5}$

$\overline{P_1}$ = Prob. That Aman will not solve = $\frac{2}{5}$

Odds in favour of Mohan = 4:1

P_2 = Prob. That Mohan will solve = $\frac{4}{5}$

$\overline{P_2}$ = Prob,. that Mohan will not solve = $\frac{1}{5}$

Odds in favour of Bipin = 2:1

P_3 = Prob. That Bipin will solve = $\frac{1}{3}$

$\overline{P_3}$ = Prob,. that Bipin will not solve = $\frac{2}{3}$

i. The prob. that all three will solve = P_1 x P_2 x P_3

$\frac{3}{5} \text{x} \frac{4}{5} \text{x} \frac{1}{3} = \frac{4}{25}$

ii. the prob. that will be solve = Prob. That atleast one solve

= 1 – (Prob. that no one solve)

= 1 - $\overline{P_1}.\overline{P_2}.\overline{P_3}$

= 1 - $\frac{2}{5} \text{x} \frac{1}{5} \text{x} \frac{2}{3} = 1 = \frac{4}{75}$

= $\frac{75-4}{75} = \frac{71}{75}$

Question: 9 [5+5=10]

a. Find the locus of the complex number z = x + iy, satisfying relations arg(z-1) = $\frac{\pi}{4}$ and $|z-2-3i| = 2$. . Illustrate the locus on the Argand plane.

Answer:

Given: Complex number, z = x + iy

arg (z-1) = $\frac{\pi}{4}$

$\Rightarrow$ arg (x + iy -1) = $\frac{\pi}{4}$

$\Rightarrow \arg[(x-1) + iy] = \frac{\pi}{4}$

$\Rightarrow \tan^{-1} \frac{y}{x-1} = \frac{\pi}{4}$

$\Rightarrow \frac{y}{x-1} = \tan\frac{\pi}{4}$

$\Rightarrow \frac{y}{x-1} = 1 \Rightarrow x - 1 = y$

$\Rightarrow x - y = 1$

$\Rightarrow x = y + 1$ ....(i)

$\Rightarrow |z-2-3i| = 2$

Given z = x + iy, then

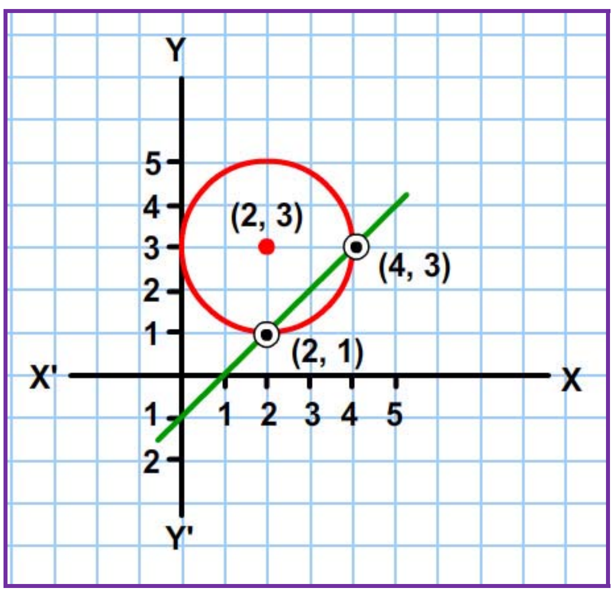
$\Rightarrow |x+iy-2-3i| = 2$

$\Rightarrow |(x-2)+i(y-3)| = 2$

$\Rightarrow \sqrt{(x-2)^2+(y-3)^2} = 2$

$\Rightarrow \left|(x-2)^2+(y-3)^2\right| = (2)^2$ .....(ii)

This equation (ii) represents a circle of radius 2 and centre (2,3).
Solving eqs. (i) and (ii), we get
y = 3,1
when y = 3, x = 4
when y = 1, x = 2
$\therefore$Locus on Argand plane are (4,3) and (2,1).



b. Solve the following differential equation:

$ye^y dx = (y^3 + 2xe^y)dy$, given that x = 0, y = 1.

Answer:
Given : $ye^y\, dx = (y^3 + 2x\, e^y)\, dy$

$$\frac{dx}{dy} = y2e^{-y} + \frac{2x}{y}, \text{ if } y \neq 0$$

$$\frac{dx}{dy} - \frac{2}{y}x = y2e^{-y}$$

It is a linear differential equation with $P = \frac{-2}{y}$ and $Q = y2e^{-y}$

$$\text{I.F.} = e^{-\int \frac{2}{y} dy} = e^{-2\log y} = e^{\log y^{-2}}$$

Now multiplying both sides of (1) by $\frac{1}{y^2}$, we get

$$\frac{1}{y^2} \cdot \frac{dx}{dy} - \frac{2}{y3}x = e^{-y}$$

Integrating both sides of (2), w.r.t. y, we get

$$x\left(\frac{1}{y^2}\right) = \int e^{-y}\, dy + c$$

$$\frac{x}{y^2} = -e^{-y} + c$$

$\because$ x = 0 and y = 1, putting this in (3), we get

$$0 = -e^{-1} + c \Rightarrow c = \frac{1}{e}$$

Putting $c = \frac{1}{e}$ in eq. (3), we get

$$\frac{x}{y^2} = -e^{-y} + \frac{1}{e}$$

$x = y^2 (e^{-1} - e^{-y})$

hence, $x = y^2 (e^{-1} - e^{-y})$, $y \neq 0$ gives the required solution.

Section B (Compulsory) *(Question numbers 10 to 12)*

Question: 10 [5+5=10]

a. If $\vec{a}$ and $\vec{b}$ are unit vectors and θ is the angle between them, then show that $\left|\vec{a}-\vec{b}\right| = 2\sin\frac{\theta}{2}$

Answer:

$\vec{a}.\vec{b} = \left|\vec{a}\right| . \left|\vec{b}\right| \cos\theta$

$= 1 \times 1 \cos\theta = \cos\theta$

$\left|\vec{a}-\vec{b}\right|^2 = (\vec{a}.\vec{b})^2$

$= a^2 + b^2 - 2\vec{a}.\vec{b}$

$= 2 - 2\left(1 - 2\sin^2\frac{\theta}{2}\right)$

$= 2 - 2 + 4\sin^2\frac{\theta}{2}$

$= 4\sin^2\frac{\theta}{2}$

Hence, $\left|\vec{a}-\vec{b}\right| = \sqrt{4\sin^2\frac{\theta}{2}}$

$\left|\vec{a}-\vec{b}\right| = 2\sin\frac{\theta}{2}$.

b. Find the value of λ for which the four points A,B, C,D with position vectors $-\hat{j}-\hat{k}; 4\hat{i}+5\hat{j}+\lambda\hat{k}; 3\hat{i}+9\hat{j}+4\hat{k}$ and $-4\hat{i}\ 4\hat{j}+4\hat{k}$ are coplanar.

Answer:

Let O be the origin and we have

$\overrightarrow{OA} = -\hat{j}-\hat{k}, \overrightarrow{OB} = 4\hat{i}+5\hat{j}+\lambda\hat{k}, \overrightarrow{OC} = 3\hat{i}+9\hat{j}+4\hat{k}$ and $\overrightarrow{OD} = 4\hat{i}+4\hat{j}+4\hat{k}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \left(4\hat{i}+5\hat{j}+\lambda\hat{k}\right) - \left(-\hat{j}-\hat{k}\right)$

$= (4\hat{i}+6\hat{j}) + (\lambda+1)\hat{k}$

$= \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \left(3\hat{i}+9\hat{j}+4\hat{k}\right) - \left(-\hat{j}-\hat{k}\right)$

$= 3\hat{i}+10\hat{j}+5\hat{k}$

and $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \left(-4\hat{i}+4\hat{j}+4\hat{k}\right) - \left(-\hat{j}-\hat{k}\right)$

$= -4\hat{i}+5\hat{j}+5\hat{k}$

$\because$ These are coplanar, therefore

$$\because \left[\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}\right] = \begin{vmatrix} 4 & 6 & \lambda+1 \\ 3 & 10 & 5 \\ -4 & 5 & 5 \end{vmatrix}$$

$\Rightarrow$ 4(50 – 25) – 6(15 + 20) + (λ+1) (15 + 40) = 0

$\Rightarrow$ 100 – 219 + 55λ + 55 = 0

$\Rightarrow$ 55λ = 55

$\Rightarrow \lambda = 1$

Question: 11 [5+5=10]

a. Find the equation of a line passing through the point (-1,3,-2) and perpendicular to the lines: $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ and $\frac{x+2}{-3}=\frac{y-1}{2}=\frac{z+1}{5}$.

Answer:

Let the direction rations of the required line be a,b,c.

Since it is perpendicular to the two given lines,

Therefore , $a + 2b + 3c = 0$ (i)

$-3a + 2b + 5c = 0$ (ii)

Solving (i) and (ii) by cross-multiplication, we get

$$\frac{a}{4} = \frac{b}{-14} = \frac{c}{8}$$

i.e., $\frac{a}{2} = \frac{b}{-7} = \frac{c}{4} = \kappa(\text{say})$

Thus the required line passed through (-1 , 3 , -2) and has direction ratios proportional to (2, -7, 4).

Hence, the equation of a line is:

$$\frac{x+1}{2} = \frac{y-3}{-7} = \frac{z+2}{4}$$

b. Find the equation of the plane parallel to the plane 2x-4y + 4z = 7 and which are at a distance of five units from the point (3,-1,-2).

Answer:

Any plane parallel to $2x - 4y + 4z = 7$ is given by

$2x - 4y + 4z + k = 0$

Its at a distance of 5 from (3, -1, 2)

$$\left|\frac{2(3)-4(-1)+4(2)+\kappa}{\sqrt{4+16+16}}\right| = 5$$

$$\left|\frac{6+4+8+\kappa}{\sqrt{36}}\right| = 5$$

$\kappa + 18 = \pm 30$

$\kappa + 18 = 30$ or $\kappa + 18 = -30$

$\kappa = 30 - 18$ or $\kappa = -30 - 18$

$\kappa = 12$ or $\kappa = -48$

$\therefore$ Equation of required planes are $2x - 4y + 12 = 0$ and $2x - 4y + 4z - 48 = 0$.

Question: 12 [5+5=10]

a. If the sum and the product of the mean and variance of a Binomial Distribution are 1.8 and 0.8 respectively, find the probability distribution and the probability of at least one success.

Answer:

Here, mean(np) + variance (npq) = 1.8 …(i)

and mean (np) x variance (npq) = 0.8

we know that

$= \sqrt{(\text{mean}+\text{variance})^2 - 4 \text{ x mean x variance}}$

$= \sqrt{(1.8)^2 - 4 \text{ x } 0.8}$

$= \sqrt{3.24 - 3.2}$

$= \sqrt{0.04}$

mean – Variance = 0.2 ...(ii)

Solving (i) and (ii), we get
mean = 1 i.e , np = 1
Variance = 0.8 i.e, npq = 0.8

$q = \frac{0.8}{1} = \frac{8}{10} = \frac{4}{5}$

$p = 1 - \frac{4}{5} = \frac{1}{5}$

np = 1 $\Rightarrow$ n. $\frac{1}{5}$ = 1 $\Rightarrow$ n = 5

Hence, distribution is $\left(\frac{1}{5}+\frac{4}{5}\right)^5$.

b. For A, B and C, the chances of being selected as the manager of a firm are 4:1:2 respectively. The probabilities for them to introduce a radical change in the marketing strategy are 0.3, 0.8 and 0.5 respectively. If a change takes place; find the probability that it is due to the appointment of B.

Answer:

Let E_1, E_2 and E_3 and F be the events defined as follows:
E_1 = taking chances by A
E_2 = taking chances by B
E_3 = taking chances by c
F = selected person as the manager of a firm.
Since there are 7 chances,

$P(E_1) = \frac{4}{7}, P(E_2) = \frac{1}{7} \text{ and } P(E_3) = \frac{2}{7}$

It is given that

$P\left(\frac{F}{E_1}\right) = 0.3, P\left(\frac{F}{E_2}\right) = 0.8 \text{ and } P\left(\frac{F}{E_3}\right) = 0.5$

By Baye's rule, we get

$$P\left(\frac{F}{E_2}\right) = \frac{P(E_2).P\left(\frac{F}{E_2}\right)}{P(E_2)P\left(\frac{F}{E_2}\right) + P(E_1)P\left(\frac{F}{E_1}\right) + P(E_3).P\left(\frac{F}{E_3}\right)}$$

$$= \frac{\frac{1}{7} \text{x} 0.8}{\frac{1}{7} \text{x} 0.8 + \frac{4}{7} \text{x} 0.3 + \frac{2}{7} \text{x} 0.5}$$

$$= \frac{\frac{0.8}{7}}{\frac{0.8+1.2+1}{7}} = \frac{0.8}{3}$$

$$= \frac{8}{30} = \frac{4}{15}$$

Section C (Statistics) *(Question numbers 13 to 15)*

Question 13

[5+5=10]

a. If Mr. Nirav deposits ₹ 250 at the beginning of each month in an account that pays an interest of 6% per annum compounded monthly, how many months will be required for the deposit to amount to atleast ₹ 6,390?

Answer:

We know that amount of annuity, M = $\frac{A}{r}$ (1+r) [(1+r)m – 1]

Where M = 6,390, A = Rs. 250, r = 6% p.a compounded monthly or r = $\frac{6}{12 \times 100}$ = 0.005

$$6390 = \frac{250}{0.005}(1.005)\ [(1 + 0.005)^n - 1\ ]$$

$$\frac{6390\ x\, 0.005}{250\ x\ 1.005} = (1 + 0.005)^n - 1$$

$\therefore$ (1..005)n = 1.1244

Taking log on both sides, we get

n log 1.005 = log 1.244

$$n = \frac{\log 1.244}{\log 1.005}$$

By using log tables, we get

$$n = \frac{0.0512}{0.0026}$$

n = 19.69

i.e n = 20 months

b. A mill owner buys two types of machines A and B for his mill. Machine A occupies 1000 sqm of area and requires 12 men to operate it; while machine B occupies 1200 sqm of area and requires 8 men to operate it. The owner has 7600 sqm of area available and 72 men to operate the machines. If machine A produces 50 units and machine B produces 40 units daily, how many machines of each type should he buy to maximize the daily output? Use Linear Programming to find the solution.

Answer:

Let x machines of type A an dy machines of type B manufactured to maximize the daily output, then the L.P.P is

Maximize, Z = 50x + 40h

Subject to $\quad 1000x + 1200y \le 7600$

$12x + 8y \le 72$

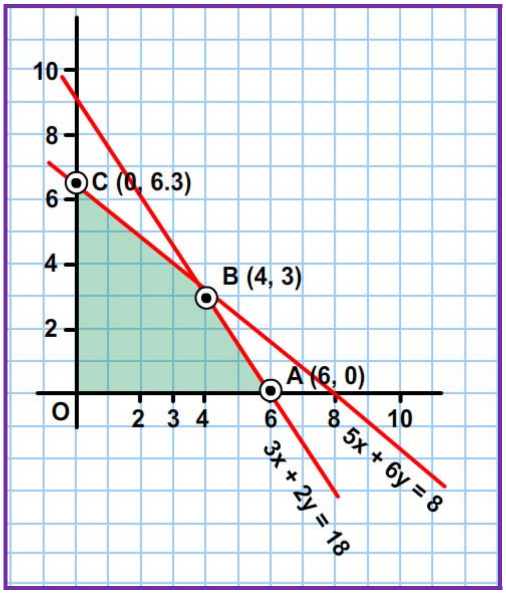
$x, y \le 0$

The coordinates of the vertices (corner point) of shaded feasible region ABC are A(6,0), B(4,3), C(0, 6.3).

These points are obtained by solving the equations of corresponding intersecting lines. The value of objective functions are:

Points	Z = 50x + 40y
A(6,0)	300
B(4,3)	320
C(0, 6.3)	253.3

Hence, 4 machines of type A and 3 machines of type B should be bought to maximize the daily output.



Question 14

[5+5=10]

a. A bill of ₹ 60,000 was drawn on 1st April 2011 at 4months and discounted for ₹ 58,560 at a bank. If the rate of interest was 12% per annum, on what date was the bill discounted?

Answer:

Given: Bill value, A = ₹Rs. 60,000

Discount value of the bill = ₹58, 560

Banker's discount = ₹ (60,000 – 58,560)

= ₹. 1,440.

Rate of interest = 12%

Banker's discount = A n 1

$$1,440 = 60,00 \text{ x } n \text{ x } \frac{12}{100}$$

1,440 = 7,200 x n

$$n = \frac{1,440}{7,200} = \frac{1}{5} \text{years}$$

$$n = \frac{1}{5} \text{x} 365 \, \text{days}$$

Therefore, August – 4 days

July – 31 days

June – 30 days

May – 8 days

Hence, the bill was discounted on 23th May.

b. A company produces a commodity with ₹ 24,000 fixed cost. The variable cost is estimated to be 25% of the total revenue recovered on selling the product at a rate of Rs. 8 per unit. Find the following:
   (i) Find the Marginal Cost (MC)
   (ii) Revenue function
   (iii) Breakeven point.

Answer:
Let x be the number of units produced.
Given: Fixed costs =₹ 24,000
$\therefore$ Revenue R(x) = ₹ 8x
Variable cost = 25% of total revenue
$= ₹\frac{25}{100} x 8x$
= ₹. 2x.
i cost function C(x) = Fixed cost + Variable cost
= 24,000 + 2x
ii. Revenue function R(x) = 8x
iii. For break even values, we should have
R(x) = C(x)
8x = 24,000 + 2x
8x – 2x = 24,000
x = 4,000
Hence, the break even value is 4,000.

Question 15 [5+5=10]

a. The price index for the year 2011 taking 2001 as the base year was 127. The simple average of price relatives method was used. Find the value of x:

Items	A	B	C	D	E	F
Price (₹. Per unit) in year 2001	80	70	50	20	18	25
Price (₹. Per unit) in year 2011	100	87.50	61	22	X	32.50

Answer:
First we tabulate the given data in the following form:

Items	Price (Rs. Per unit) in year 2001	Price (Rs. Per unit) in year 2011	Price relatives $\frac{p_1}{p_0} \text{x} 100$
A	80	100	$\frac{100}{80} \text{x} 100 = 125$
B	70	87.50	$\frac{87.50}{70} \text{x} 100 = 125$
C	50	61	$\frac{61}{50} \text{x} 100 = 122$
D	20	22	$\frac{22}{20} \text{x} 100 = 110$
E	18	x	$\frac{x}{18} \text{x} 100 = \frac{100x}{18}$
F	25	32.50	$\frac{32.50}{25} \text{x} 100 = 130$
	$\Sigma p_0 = 263$	$\Sigma p_1 = 303 + x$	$\Sigma \frac{p_1}{p_0} \text{x} 100 = 612 + \frac{100x}{18}$

Given: Index number = 127

Price Index P_{01} = $P_{01} = \frac{1}{N}\left(\Sigma \frac{P_1}{P_0} \text{x} 100\right)$

$$127 = \frac{1}{6}\left(612 + \frac{100x}{18}\right)$$

$$127 = \frac{11{,}016 + 100x}{18}$$

$\Rightarrow$ 127 + 18 = 11,016 + 100x
$\Rightarrow$ 13716 – 11,016 = 100x
2700 = 100x
x = 27

b. The profit of paper bag manufacturing company (in lakhs of rupees) during each month of a year are:

Month	Jan	Feb	Mar	Apr	May	June	July	Aug	Sept	Oct	Nov	Dec
Profit	1.2	0.8	1.4	1.6	2.0	2.4	3.6	4.8	3.4	1.8	0.8	1.2

Plot the given data on a graph sheet. Calculate the four monthly moving averages and plot these on the same graph sheet. [5]

Answer:

Month	Profit	4 Monthly moving	4 years moving average	4 years moving average centred
Jan	1.2			
Feb	0.8	5.0	1.25	1.35
March	1.4	5.8	1.45	1.65
April	1.6	7.4	1.85	2.125
May	2.0	9.6	2.4	2.8
June	2.4	12.8	3.2	3.375
July	3.6	14.2	3.55	3.475
August	4.8	13.6	3.4	3.05
September	3.4	10.8	2.7	2.25
October	1.8	7.2	1.8	
November	0.8			
December	1.2			

The corresponding histogram and the 4 years moving average are shown in the graph drawn below: