2014

Section A *(Question numbers 1 to 9)*

Question: 1

[3 x 10 = 30]

i. If A = $\begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$, find the values of x and y such that $A^2 + xI_2 = yA$.

Answer:

$$A = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 16 & 8 \\ 56 & 32 \end{bmatrix}$$

$$A^2 + xI_2 = yA$$

$$\Rightarrow \begin{bmatrix} 16 & 8 \\ 56 & 32 \end{bmatrix} + x\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = y\begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$$

$\therefore$ x = 8 and y = 8

ii. Find the eccentricity and the coordinates of foci of the hyperbola $25x^2 - 9y^2 = 225$

Answer:

$$\frac{x^2}{9} - \frac{y^2}{25} = 1$$

$$a^2 = 9, b^2 = 25$$

$$b^2 = a^2\left(e^2 - 1\right)$$

$$\Rightarrow \frac{25}{9} = e^2 - 1$$

$$\Rightarrow e^2 = \frac{34}{9}$$

$$e = \frac{\sqrt{34}}{3}$$

$$F\left(\pm\sqrt{34}, 0\right)$$

iii. Evaluate : $\tan\left[2\tan^{-1}\frac{1}{2} - \cot^{-1}3\right]$

Answer:

$$\tan\left[2\tan^{-1}\frac{1}{2} - \cot^{-1}3\right]$$

$$= \tan\left[\tan^{-1}\frac{1}{1-\frac{1}{4}} - \tan^{-1}\frac{1}{3}\right]$$

$$= \tan\left[\tan^{-1}\frac{4}{3} - \tan^{-1}\frac{1}{3}\right]$$

$$= \tan\left[\tan^{-1}\frac{\frac{4}{3}-\frac{1}{3}}{1+\frac{4}{3}\times\frac{1}{3}}\right]$$

$$= \frac{1}{\frac{13}{9}}$$

$$= \frac{9}{13}$$

iv. Using L 'Hospital's' Rule , Evaluate : $\lim\limits_{x\to 0}(1+\sin x)^{\cot x}$

Answer:

$$\text{Let } A = \lim_{x\to 0}(1+\sin x)^{\cot x}$$

$$\text{Log}\, A = \mathop{\text{Lt}}_{x\to 0} \cot x \log(1+\sin x)$$

$$= \mathop{\text{Lt}}_{x\to 0} \frac{\log(1+\sin x)}{\tan x}$$

$$= \mathop{\text{Lt}}_{x\to 0} \left(\frac{\frac{\text{Cos}\, x}{1+\sin x}}{\sec^2 x}\right)$$

= 1

$\Rightarrow A = e$

OR Log A = 1

v. Evaluate : $\int e^x \frac{(2+\sin 2x)}{\cos^2 x}dx$

Answer:

$$\int e^x \frac{(2+\sin 2x)}{\cos^2 x}dx$$

$$= \int e^x \left(2\sec^2 x + 2\tan x\right)dx$$

$$= 2\int e^x \sec^2 x dx + 2\left(\tan x\, e^x - \int \sec^2 x\, e^x dx\right)$$

$$= 2e^x \tan x + c$$

vi. Using properties of definite integrals, evaluate : $\int_0^{\frac{x}{3}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$

Answer:

$$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx \quad \text{......eqn. (1)}$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}dx$$

$$I=\int_0^{\frac{\pi}{2}}\frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx \quad \text{.......eqn. (2)}$$

Adding eqn. (1) and eqn. (2)

$$2I=\int_0^{\frac{\pi}{2}}1\,dx$$

$$2I = (x)_0^{\frac{\pi}{2}}$$

$$2\,I = \frac{\pi}{2}$$

$$\therefore I=\frac{\pi}{4}$$

vii. For the given lines of regression, $3x - 2y = 5$ and $x - 4y = 7$, find :

Answer:

Let the line of regression of x on y be $3x - 2y = 5$ and y on x be $x - 4y = 7$.

Writing the first equation in the form $x=\frac{2}{3}y+\frac{5}{3}$, we get $b_{xy}=\frac{2}{3}$

Writing the second equation in any form $y=\frac{1}{4}x-\frac{7}{4}$, we get $b_{yx}=\frac{1}{4}$

Now $r^2=b_{yx}\times b_{xy}$

$$r^2=\frac{1}{4}\times\frac{2}{3}$$

$$r^2=\frac{1}{6}$$

$r=\frac{1}{\sqrt{6}}$. since $r.b_{yx}$ and b_{xy} all have same sign .

a. Regression coefficients b_{yx} and b_{xy}

Answer:

b. Coefficient of correlation $r(x, y)$

Answer:

viii. Express the complex number $\frac{\left(1+\sqrt{3}i\right)^2}{\sqrt{3}-i}$ in the form of $a + ib$. Hence, find the modulus and argument of the complex number.

Answer:

$$\frac{\left(1+\sqrt{3}\,i\right)}{\sqrt{3}-i}=\frac{1-3+2\sqrt{3}\,i}{\sqrt{3}-i}$$

$$= \frac{-2+2\sqrt{3}i}{\sqrt{3}-i} \times \frac{\sqrt{3}+i}{\sqrt{3}+i}$$
$$= \frac{4\sqrt{3}+4i}{3+1}$$
$$= -\sqrt{3}+i$$

Modulus = $\sqrt{3+1}$ = 2

Argument = $\tan^{-1}\frac{-1}{\sqrt{3}} = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

ix. A bag contains 20 balls numbered from 1 to 20. One ball is drawn at random from the bag. What is the probability that the ball drawn is marked with a number which is multiple of 3 or 4?

Answer:

P (No of multiple by 3 or 4) = $\frac{6}{20} + \frac{5}{20} - \frac{1}{20}$

$$= \frac{10}{20}$$
$$= \frac{1}{2}$$

x. Solve the differential equation: (x + 1) dy – 2 xy dx = 0

Answer:

$$(x+1)dy = 2xy\,dx$$
$$\int \frac{dy}{y} = \int \frac{2x}{x+1} dx$$
$$\Rightarrow \text{Log}\, y = 2\int \frac{x+1-1}{+1} dx = 2\int \left(1 - \frac{1}{x+1}\right) dx$$
$$\Rightarrow \text{Log } y = [x - \log(x+1)] + \text{Log}\, c$$
$$\Rightarrow \text{Log} \frac{y(x+1)^2}{c} = 2x$$
$$\frac{y(x+1)^2}{c} = e^{2x}$$
$$y = \frac{c\,e^{2x}}{(x+1)^2}$$

Question: 2

[5+5=10]

i. Using properties of determinants, prove that :

$$\begin{vmatrix} a^2+1 & ab & ac \\ ba & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = a^2+b^2+c^2+1$$

Answer:

$$\begin{vmatrix} a^2+1 & ab & ac \\ ba & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix}$$

Taking a, b ,c common from R_1, R_2, R_3

$$= abc\begin{vmatrix} a+\frac{1}{a} & ab & c \\ a & b+\frac{1}{b} & c \\ a & b & c+\frac{1}{c} \end{vmatrix}$$

$$R_1 \to R_1 - R_2, \quad R_2 \to R_2 - R_3$$

$$= abc\begin{vmatrix} \frac{1}{a} & -\frac{1}{b} & 0 \\ 0 & \frac{1}{b} & -\frac{1}{c} \\ a & b & c+\frac{1}{c} \end{vmatrix}$$

Taking $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ common from C_1, C_2, C_3 respectively.

$$= \frac{abc}{abc}\begin{vmatrix} 1 & -1 & 0 \\ 0 & \frac{1}{b} & -\frac{1}{c} \\ a^2 & b^2 & c^2+1 \end{vmatrix}$$

$$= \left(c^2+1+b^2\right)+a^2$$

$$= a^2+b^2+c^2+1$$

i. Using matrix method , solve the following system of equation:
$x - 2y = 10$, $2x + y + 3z = 8$ and $-2y + z = 7$

Answer:

$$A = \begin{pmatrix} 1 & -2 & 0 \\ 2 & 1 & 3 \\ 0 & -2 & 1 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 1 & -2 & 0 \\ 2 & 1 & 3 \\ 0 & -2 & 1 \end{vmatrix}$$

= 11

$$A^{-1} = \frac{1}{11}\begin{pmatrix} 7 & 2 & -6 \\ -2 & 1 & -3 \\ -4 & 2 & 5 \end{pmatrix}$$

$$X = A^{-1}B$$

$$X = \frac{1}{11}\begin{pmatrix} 7 & 2 & -6 \\ -2 & 1 & -3 \\ -4 & 2 & 5 \end{pmatrix}\begin{pmatrix} 10 \\ 8 \\ 7 \end{pmatrix}$$

$$= \frac{1}{11}\begin{pmatrix} 44 \\ -33 \\ 11 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$$

$\therefore$ x = 4, y= -3, z = 1

Question: 3 [5+5=10]

i. If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$, prove that : $x^2 + y^2 + z^2 + 2xyz = 1$

Answer:

LHS

$\cos^{-1}x + \cos^{-1} + \cos^{-1}z = \pi$

$\Rightarrow \cos^{-1}x + \cos^{-1}y = \pi - \cos^{-1}z$

$\Rightarrow \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right) = \pi - \cos^{-1}z$

$\Rightarrow xy - \sqrt{1-x^2}\sqrt{1-y^2} = \cos\left(\pi - \cos^{-1}z\right)$

$\Rightarrow xy - \sqrt{1-x^2}\sqrt{1-y^2} = -\cos\cos^{-1}z$

$\Rightarrow xy - \sqrt{1-x^2}\sqrt{1-y^2} = -z$

$\Rightarrow xy + z = \sqrt{1-x^2}\sqrt{1-y^2}$

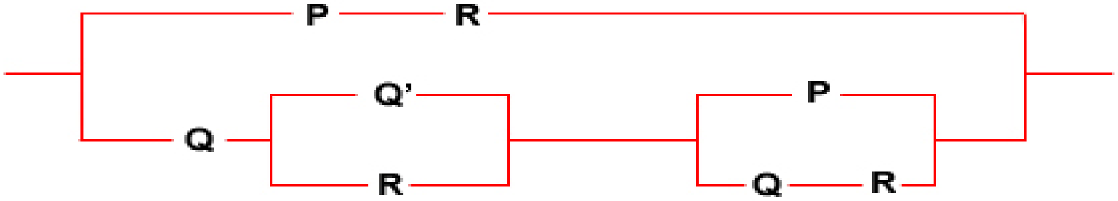
Squaring both sides

$\Rightarrow x^2y^2 + z^2 + 2xy = 1 - x^2 - y^2 + x^2y^2$

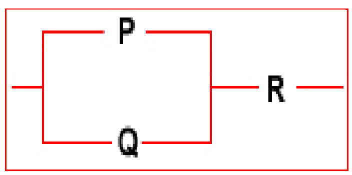
$\Rightarrow x^2 + y^2 + z^2 + 2xyz = 1$

ii. P, Q and R represent switches in on position a P',Q' represent switches in off position. Construct a switching circuit representing the polynomial PR + Q(Q' + R)(P + QR). Using Boolean algebra, simplify the polynomial expression and construct the simplified circuit.

Answer:



= PR + Q(Q' + R)(P + QR)
= PR +(QQ' + QR)(P + QR)
= PR + QR(P + QR) $\quad (\because\ QQ' = 0)$
= PR + PQR + QR $\quad (\because\ 1 + Q = 1)$
= PR (1 + Q) + QR
= PR + QR
= R (P + Q)
∴ Simplified circuit will be



Question: 4 [5+5=10]

i. Verify Rolle's Theorem for the function f(x) = e^x(sin x - cos x) on $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$

Answer:

$f(x) = e^x(\sin x - \cos x)$ is continuos in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$

$\therefore f(x)$ is diffrentable in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$

$f'(x) = e^x(\cos x + \sin x) + e^x(\sin x - \cos x) = 2e^x \sin x$

$f(x)$ exists in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$

$f\left(\frac{\pi}{4}\right) = \left(\frac{5\pi}{4}\right) = 0$

There exist 'c' in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$ such that $f'(c) = 0$

$2e^c \sin c = 0,\ c = 0,\ \pi$
$\Rightarrow c = x$

c lies between $\frac{\pi}{4}$ and $\frac{5\pi}{4}$.
Hence, Rolle's theorem is verified

ii. Find the equation of the parabola with latus-rectum joining points (4,6) and (4, - 2).

Answer:

Question: 5

[5+5=10]

i. If $y=\frac{x\sin^{-1}x}{\sqrt{1-x^2}}$, prove that : $\left(1-x^2\right)\frac{dy}{dx}=x+\frac{y}{x}$

Answer:

$$\frac{dy}{dx}=\frac{\left(\sin^{-1}x+\frac{x}{\sqrt{1-x^2}}\right)\sqrt{1-x^2}-x\sin^{-1}x\left(\frac{-2x}{2\sqrt{1-x^2}}\right)}{1-x^2}$$

$$\Rightarrow\left(1-x^2\right)\frac{dy}{dx}=\sqrt{1-x^2}\sin^{-1}x+x+xy$$

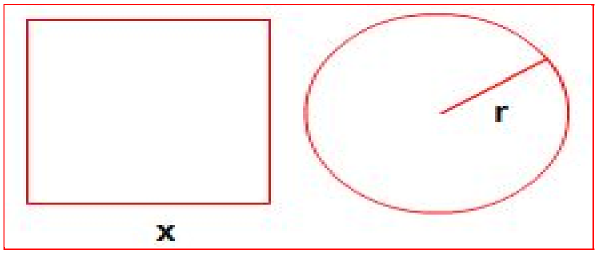
$$\Rightarrow\left(1-x^2\right)\frac{dy}{dx}=\left(1-x^2\right)\frac{y}{x}+x+xy$$

$$\Rightarrow\left(1-x^2\right)\frac{dy}{dx}=\frac{y}{x}-xy+x+xy$$

$$\Rightarrow\left(1-x^2\right)\frac{dy}{dx}=x+\frac{y}{x}$$

ii. A wire of length 50 m is cut into two piece. One piece of the wire bent in the shape of a square and the other in shape of circle. What should be the length of each piece so that the combined area of the two is minimum?

Answer:



Let the side of the square be x and radius of the circle r.

$$4x+2\pi r=50$$
$$2\pi r=50-4x$$
$$\Rightarrow r=\frac{50-4x}{2\pi}$$
$$A=x^2+\pi r^2$$
$$=x^2+\pi\left(\frac{50-4x}{2\pi}\right)^2$$
$$=x^2+\frac{1}{4\pi}(50-4x)^2$$

For $\min^m$ A, $\frac{dA}{dx}=0$

$$\frac{dA}{dx}=2x+\frac{1}{4\pi}2(50-4x)(-4)=0$$

$\Rightarrow 2\pi x - 100 + 8x = 0$

$\Rightarrow x = \dfrac{50}{\pi + 4}$

$\dfrac{d^2A}{dx^2} = +ve$, A is min^m at $x = \dfrac{50}{\pi + 4}$

$\therefore$ Length of square $= \dfrac{200}{\pi + 4}$

Length of circle = $50 - \dfrac{200}{\pi + 4} = \dfrac{50\pi}{\pi + 4}$

Question: 6 [5+5=10]

i. Evaluate: $\int \dfrac{x + \sin x}{1 + \cos x}\, dx$

Answer:

$$I = \int \frac{x + \sin x}{1 + \cos x} dx$$

$$= \int \frac{x + \sin x}{2\cos^2 \frac{x}{2}} dx$$

$$= \int \left(\frac{x}{2\cos^2 \frac{x}{2}} + \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}} \right) dx$$

$$= \int \left(\frac{1}{2} x \sec^2 \frac{x}{2} - \tan\frac{x}{2} \right) dx$$

$$= \frac{1}{2} \int x \sec^2 \frac{x}{2} dx + \int \tan\frac{x}{2} dx$$

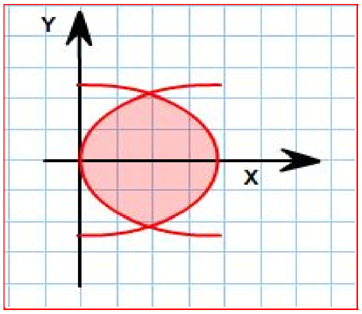
$$= \frac{1}{2} \left[x \frac{\tan\frac{x}{2}}{\frac{1}{2}} - \int \left(\frac{\tan\frac{x}{2}}{\frac{1}{2}} \right).1\, dx \right] + \int \tan\frac{x}{2}\, dx + c$$

$$= x\tan\frac{x}{2} - \int \tan\frac{x}{2}\, dx + \int \tan\frac{x}{2} dx + c$$

$$= x\tan\frac{x}{2} + c$$

ii. Sketch the graphs of the curves $y^2 = x$ and $y^2 = 4 - 3x$ and find the area enclosed between them.

Answer:



$y^2 = 4 - 3x$ .... (1)

$y^2 = x$ .... (2)

Solving (1) & (2) we get x = 4 – 3x
4x = 4 , x = 1

Required Area = $= 2\left[\int_0^1 \sqrt{x}\,dx + \int_1^{\frac{4}{3}} \sqrt{4-3x}\,dx\right]$

$= 2\left(\frac{2}{3} + \frac{2}{9}\right)$

$= \frac{16}{9}$ sq units.

Question: 7 [5+5=10]

i. A psychologist selected a random sample of 22 students. He grouped them in 11 pairs so that the students in each pair have nearly equal scores in an intelligence test. In each pair, one student was taught by method A and the other by method B and examined after the course. The marks obtained by them after the course are as follows :

Pairs	1	2	3	4	5	6	7	8	9	10	11
Method A	24	29	19	14	30	19	27	30	20	28	11
Method B	37	35	16	26	23	27	19	20	16	11	21

Calculate Spearman's Rank correction.

Answer:

Pairs	A	B	Rank A	Rank B	D	D^2
1	24	37	6	1	5	25
2	29	35	3	2	1	1
3	19	16	8.5	9.5	1	1
4	14	26	10	4	6	36
5	30	23	1.5	5	-3.5	12.25

6	19	27	8.5	3	5.5	30.25
7	27	19	5	8	-3	9
8	30	20	1.5	7	-5.5	30.25
9	20	16	7	9.5	-2.5	6.25
10	28	11	4	11	-7	49
11	11	21	11	6	5	25
						$\sum d^2 = 225$

$$R = 1 - \frac{6\left[\sum d^2 + \sum \frac{\left(n^3 - n\right)}{12}\right]}{n\left(n^2 - 1\right)}$$

$$= 1 - 6\left[225 + \frac{\left(2^3 - 2\right)}{12} + \frac{\left(2^3 - 2\right)}{12} + \frac{\left(2^3 - 2\right)}{12}\right]\left(11 \times 120\right)$$

$$= 1 - \frac{6\left(225 + 1.5\right)}{11 \times 120}$$

$$= 1 - \frac{1350}{1320}$$

= 1 – 1.029

= - 0.029,

Shows a negative low correlation.

ii. The coefficient of correlation between the values denoted by X and Y is 0.5. The mean of X is 3 and that of Y is 5. Their standard deviations are 5 and 4 respectively. Find

a. The two lines of regression

Answer:

$\overline{X} = 3,\ \overline{Y} = 5,\ \sigma_x = 5,\ \sigma_y = 4,\ r = 0.5$

The line of regression of Y and X is given by $y - \overline{Y} = r\frac{\sigma_y}{\sigma_x}\left(x - \overline{X}\right)$

$$y - 5 = 0.5 \times \frac{4}{5}\left(x - 3\right)$$

5y – 25 = 2x – 6

2x – 5y + 19 = 0 …………….. eqn (1)

The line of regression of X on Y is given by $x - \overline{X} = r\frac{\sigma_x}{\sigma_y}\left(y - \overline{Y}\right)$

$x-3=0.5\times\frac{5}{4}(y-5)$

8x – 24 = 5y – 25

8x – 5y + 1 = 0 ……............. ..eqn(2)

b. The expected value of Y, when X is given 14

Answer:

Expected value of y when x = 14

2 x 14 – 5y +19 = 0

$\Rightarrow y=9.4$

c. The expected value of X, When Y is given 9.

Answer:

8x – 5y +1 = 0

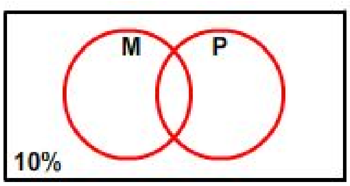
$\Rightarrow x=5.5$

Question: 8 [5+5=10]

i. In a College, 70% students pass in Physics, 75% pass in Mathematics and 10% student fail in both. One student is chosen at random. What is the probability that :

a. He passes in Physics and Mathematics.

Answer:



P(M) = 0.75

P(P) = 0.70

P(MUP) = 0.90

$P(MUP)=P(M)+P(P)-P(M\cap P)$

$0.90=0.75+0.70-P(M\cap P)$

$\Rightarrow P(M\cap P)=0.55$

b. He passes in Mathematics given that he passes in Physics

Answer:

$P\left(\frac{M}{P}\right)=\frac{P(M\cap P)}{P(P)}$

$P\left(\frac{M}{P}\right)=\frac{0.55}{0.70}$

$$P\left(\frac{M}{P}\right)=\frac{11}{14}$$

c. He passes in Physics given that he passes in Mathematics.

Answer:

$$P\left(\frac{P}{M}\right)=\frac{P(M\cap P)}{P(M)}$$

$$P\left(\frac{P}{M}\right)=\frac{0.55}{0.75}$$

$$P\left(\frac{P}{M}\right)=\frac{11}{15}$$

ii. A bag contains 5 white and 4 black balls and another bag contains 7 white and 9 black balls. A ball is drawn from the first bag and two balls from the second bag. What is the probability of drawing one white and two black balls?

Answer:

Bag 1 $\rightarrow$ 5W 4B

Bag 2 $\rightarrow$ 7W 9B

$$P(E)=\frac{5}{9}\times\frac{{}^{9}C_{2}}{{}^{16}C_{2}}+\frac{4}{9}\times\frac{{}^{7}C_{1}\times{}^{9}C_{1}}{{}^{16}C_{2}}$$

$$P(E)=\frac{5}{9}\times\frac{9}{2\times15}+\frac{4}{9}\times\frac{7\times9}{8\times15}$$

$$P(E)=\frac{1}{6}+\frac{7}{30}$$

$$P(E)=\frac{12}{30}$$

$$P(E)=\frac{2}{5}$$

Question: 9

[5+5=10]

i. Using De Moivre's theorem find the least positive integer n such that $\left(\frac{2i}{1+i}\right)^n$ is a positive integer.

Answer:

We have $\frac{2i}{1+i}=\frac{2i}{1+i}\times\frac{1-i}{1-i}=\frac{2(i+1)}{2}=1+i$

$$=\sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$$

$$\therefore\left(\frac{2i}{1+i}\right)^n=\left(\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right)^n$$

$$\therefore\left(\frac{2i}{1+i}\right)^n=2^{\frac{n}{2}}\left(\cos\frac{n\pi}{4}+i\sin\frac{n\pi}{4}\right)$$

Which is positive integer

If $\frac{n\pi}{4} = 0, 2\pi, 4\pi, 6\pi$
$\rightarrow$ n = 0, 8, 16, 24
$\rightarrow$ the least value of n is 8.

ii. Solve the following differential equation : $\left(3xy + y^2\right)dx + \left(x^2 + xy\right)dy = 0$

Answer:

$$\frac{dy}{dx} = -\frac{\left(3xy + y^2\right)}{\left(x^2 + xy\right)}$$

Put y = vx

$$\Rightarrow v + \frac{dv}{dx} = \frac{-\left(3v + v^2\right)}{\left(1 + v\right)}$$

$$\Rightarrow x\frac{dv}{dx} = 1 - \frac{3v + v^2}{v + 1} - v$$

$$\frac{\int 2\left(v + 1\right)dv}{\left(v^2 + 2v\right)} = -\int \frac{dx}{x}$$

$$\log\left|v^2 + 2v\right| = -\,4\log x + \log c$$

$$\log\left|v^2 + 2v\right| = \log\frac{c}{x^4}$$

$$\log\frac{\left(y^2 + 2xy\right)}{x^2} = \log\frac{c}{x^4}$$

$$y^2 + 2xy = \frac{c}{x^2}$$

Section B (Question numbers 10 to 12)

Question: 10 [5+5=10]

i. In a triangle ABC, using vectors , prove that $c^2 = a^2 + b^2 - 2ab\cos C$

Answer:

$\vec{a} + \vec{b} + \vec{c} = 0$

$\vec{a} + \vec{b} = -\vec{c}$

$\left(\vec{a} + \vec{b}\right)^2 = \vec{c}^{\,2}$

$b^2 + a^2 + 2\vec{a}.\vec{b} = c^2$

$a^2 + b^2 + 2ab\cos(\pi - c) = c^2$

$a^2 + b^2 - 2ab\cos C = c^2$

ii. Prove that : $\vec{a}.\left(\vec{b}+\vec{c}\right)\times\left(\vec{a}+2\vec{b}+3\vec{c}\right) = \left[\vec{a}\vec{b}\vec{c}\right]$

Answer:

LHS $= \vec{a}\left(\vec{b}+\vec{c}\right)\times\left(\vec{a}+2\vec{b}+3\vec{c}\right)$

$= \vec{a}.\left[\vec{b}\times\vec{a}+\vec{b}\times 2\vec{b}+\vec{b}\times 3\vec{c}+\vec{c}\times\vec{a}+c\times 2\vec{b}+\vec{c}\times 3\vec{c}\right]$

$= \vec{a}\left[\vec{b}\times\vec{a}+3\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+2\vec{c}\times\vec{b}\right]$

$= \vec{a}.\left(\vec{b}\times\vec{a}\right)+\vec{a}\left(3\vec{b}\times\vec{c}\right)+\vec{a}\left(\vec{c}\times\vec{a}\right)+\vec{a}\left(\vec{c}\times\vec{b}\right)$

$= 0 + 3\left[\vec{a}\vec{b}\vec{c}\right] + 0 - 2\left[\vec{a}\vec{b}\vec{c}\right]$

$= \left[\vec{a}\vec{b}\vec{c}\right]$

Question: 11 [5+5=10]

i. Find the equation of a line passing through the points P (-1, 3, 2) and Q (-4, 2, -2). Also, if the point R(5, 5, λ) is collinear with the points P and Q, then find the value of λ .

Answer:

The equation of a line passing through two points P(-1, 3, 2) and Q(-4, 2, -2) is given by

$$\Rightarrow \frac{x+1}{-4+1} = \frac{y-3}{2-3} = \frac{z-2}{-2-2}$$

$$\Rightarrow \frac{x+1}{-3} = \frac{y-3}{-1} = \frac{z-2}{-4}$$

OR

$$\frac{x+1}{3} = \frac{y-3}{1} = \frac{z-2}{4}$$

Now, If the three points P(-1, 3, 2), Q(-4, 2, -2) and R(5, 5, λ) are collinear, then the coordinates of point R must satisfy the equation (1)

$$\frac{5+1}{3} = \frac{5-3}{1} = \frac{\lambda-2}{4}$$

$$\Rightarrow \frac{2}{1} = \frac{\lambda - 2}{4}$$
$$\Rightarrow \lambda - 2 = 8$$
$$\Rightarrow \lambda = 10$$

ii. Find the equation of the plane passing through the points (2, -3, 1) and (-1, 1, -7) and perpendicular to the plane x – 2y + 5z + 1 = 0.

Answer:

Equation of plane passing trough (2, - 3, 1)

a(x - 2) + b(y + 3) + c(z – 1) = 0

$a(-1-2)+b(1+3)+c(-7-1)=0$

$-3a+4b-8c=0$

The given plane is perpendicular to x – 2y + 5z = 0

a – 2b + 5c = 0

Solving 1 and 2 we get a = 4k, b = 7k, c = 2k

$4(x-2)+7(y+3)+2(z-1)=0$

4x + 7y + 2z = 0

Question: 12 [5+5=10]

i. In a bolt factory, three machines A, B and C manufacture 25%, 35%, and 40% of the total production respectively. Of their respective outputs, 5%, 4% and 2% are defective. A bolt is drawn at random from the total production and it is found to be defective. Find the probability that it was manufactured by machine C.

Answer:

$P(A)=\frac{1}{4},\ P(B)=\frac{7}{20},\ P(C)=\frac{2}{5}$

Let D be the probability of drawing a defective bolt.

$P\left(\frac{D}{A}\right)=\frac{1}{20},\ P\left(\frac{D}{B}\right)=\frac{1}{25},\ P\left(\frac{D}{C}\right)=\frac{1}{50}$

$$\Rightarrow P\left(\frac{C}{D}\right)=\frac{P(C)\times P\left(\frac{D}{C}\right)}{P(A)P\left(\frac{D}{A}\right)+P(B)P\left(\frac{D}{B}\right)+P(C)P\left(\frac{D}{C}\right)}$$

$$\Rightarrow P\left(\frac{C}{D}\right)=\frac{\frac{2}{5}\times\frac{1}{50}}{\left(\frac{1}{4}\times\frac{1}{20}+\frac{7}{20}\times\frac{1}{25}+\frac{2}{5}\times\frac{1}{50}\right)}$$

$$\Rightarrow \frac{16}{69}$$

ii. On dialing certain telephone numbers, assume that on an average, One telephone number out of five is busy. Ten telephone numbers are randomly selected and dialed. Find the probability that at least three of them will be busy.

Answer:

$p=\frac{1}{5},\ q=\frac{4}{5}, n=10$

$P(x\geq 3)=1-P(x=0,1 2)$

$=1-\left[{}^{10}C_0\left(\frac{1}{5}\right)^0\left(\frac{4}{5}\right)^9+{}^{10}C_1\left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^9+{}^{10}C_2\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^8\right]$

$=1-\left[\left(\frac{4}{5}\right)^{10}+10\times\frac{1}{5}\times\left(\frac{4}{5}\right)^9+45\times\frac{1}{25}\times\left(\frac{4}{5}\right)^8\right]$

$=1-\left(\frac{4}{5}\right)^8\left[\frac{16}{25}+10\times\frac{4}{25}+\frac{4}{25}\right]$

$= 1 - 0.678$

$= 0.322$

Section C *(Question numbers 13 to 15)*

Question: 13 [5+5=10]

i. A person borrows ₹68,962 on the condition that he will repay the money with compound interest at 5% per annum in 4 equal annual instalments, the first one being payable at the end of the first year. Find the value of each installment.

Answer:

Let the installment be α .

The money borrowed is ₹68,962 which is present worth of the annuity of α payable

For 4 years (payable annually) at 5% per annum.

$\therefore$ P = ₹68962, n = 4 and $i = \frac{5}{100} = \frac{1}{20} = 0.05$

Now, $P = \frac{\alpha}{i}\left\{1-(1+i)^{n}\right\}$

$\Rightarrow 68962 = \frac{\alpha}{0.05}\left\{1-(1.05)^{-4}\right\}$

Let $x = (1.05)^{-4}$

$\therefore$ log x = - 4 log 1.05 = - 4 x 0.0212

= - 0.0848 = - 1 + 1 – 0.0848

= 1.9152

$\therefore$ x = antilog 1.9152 = 0.8226

$\therefore 68962 = \frac{\alpha}{0.05}(1-0.8226)$

$= \frac{\alpha}{0.05}\{0.1774\}$

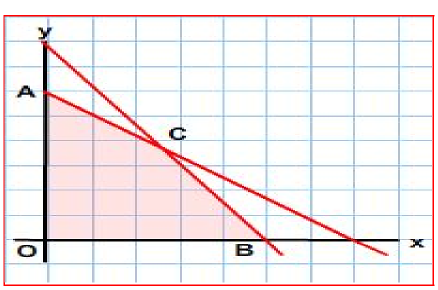
$\therefore \alpha = \frac{68962 \times 0.05}{0.1774}$

$\Rightarrow \alpha = 19436.87$

Therefore, required amount of each installment = ₹19436.87

ii. A company manufactures two types of toys A and B. A toy of type A requires 5 minutes for cutting and 10 minutes of assembling. A toy of type B requires 8 min of cutting and 8 minutes of assembling. There are 3 hrs. available for cutting and 4 hrs. available for assembling the toys in one day. The profit is ₹ 50 each on a toy of type A and ₹ 60 each on a toy of type B. How many toys of each type should the company manufacture in a day to maximize the profit? Use linear programming to find the solution.

Answer:



Let x units of type A toys and y units of type B toys.
Maximize Z = 50x + 60y
Subject to constrains

$$\begin{Bmatrix} 5x+8y \le 180 \\ 10x+8y \le 240 \end{Bmatrix}$$
x > 0, y > 0

Solving graph, we get A (0, 22.5), B (24, 0), C (12, 15)
At A, z = $50\times0+60\times22.5$ = ₹1350
B, z = $50\times24+60\times0$ = ₹1200
C, z = $50\times12+60\times12$ = ₹ 1500

Maximum profit is Rs 1500, when 12 units of type A and 15 units of type B is produced.

Question: 14 [5+5=10]

i. A firm has the cost function $C=\frac{x^3}{3}-7x^2+111x+50$ and demand function x = 100 – p. Write the total revenue function in terms of x

Answer:
Total revenue function = x(10 – x)

a. Formulate the total profit function P in terms of x.

Answer:
Profit function = Revenue function - cost function

$$=100x-x^2-\left(\frac{x^3}{3}-7x^2+111x+50\right)$$

$$=\frac{x^3}{3}+6x^2-11x-50$$

$$\frac{dP}{dx}=-x^2+12x-11$$

b. Find the profit maximizing level of output x.

Answer:
For maximum or minimum $\frac{dP}{dx}=0$

$-x^2+12x+11=0$
x = 1, 11

$$\frac{d^2P}{dx^2}=-2x+12$$
$=-2\times11+12$
$=-10<0$
Profit is maximum when x = 11

ii. A bill of ₹5050 is drawn on 13th April 2013. It was discounted on 4th July 2013 at 5% per annum. If the banker's gain on the transaction is ₹0.50. Find the nominal date of the maturity of the bill.

Answer:

Let the unexpired period of bill at the time of discounting be t years.

$B.G = \frac{A(ni)^2}{1+ni}$, where A is the face value of the bill.

Here, A = 5050 , i = 0.05

$\Rightarrow 0.50 = \frac{5050(0.05t)^2}{1+0.05t}$

$\Rightarrow 505t^2 - t - 20 = 0$

$\Rightarrow t = \frac{1 \pm \sqrt{1+40400}}{1010}$

$\Rightarrow t = \frac{1 \pm 201}{1010}$

$\Rightarrow t = \frac{1}{5} \quad (\text{neglecting} - \text{ve value})$

$\Rightarrow t = \frac{1}{5}$

$\Rightarrow t = 73 \text{ days}$

Thus, legal due date of maturity is 73 days after 4th July, which comes to 15 September.
27 days in July + 31 days in August + 15 days in September
Therefore, the legal due date is 15th September and the nominal due date is 12th September.

Question: 15 [5+5=10]

i. The price of six different commodities for years 2009 and year 2011 are as follows:

Commodities	**A**	**B**	**C**	**D**	**E**	**F**
Price in 2009 (₹)	35	80	25	30	80	x
Price in 2011(₹)	50	y	45	70	120	105

The Index number for the year 2011 taking 2009 as the base for the above data was calculated to be 125. Find the values of x and y if the total price in 2009 is ₹360.

Answer:

Commodities	Price in 2009 (in ₹)	Price in 2011 (in ₹)
A	35	50
B	80	y
C	25	45
D	30	70
E	80	120
F	x	105
	$\sum P_0 = 250 + x$	$\sum P_1 = 390 + y$

Given, $\sum P_0 = 360$
$\Rightarrow 250 + x = 360$
$\Rightarrow x = 110$

$$\text{Price index} = \frac{\sum P_1}{\sum P_0} \times 100$$

$$\Rightarrow 125 = \frac{390 + y}{360} \times 100$$

$4500 = 3900 + 10y$

Therefore, y = 60.

ii. The number of road accident in the city due to rash driving, over a period of 3 years is given in the following table.

Year	Jan-Mar	April – June	July – Sept	Oct – Dec
2010	70	60	45	72
2011	79	56	46	84
2012	90	64	45	82

Calculate four quarterly moving averages and illustrate them and original figures on the graph using the same axes for both.

Answer:

Calculation for trend by four quarterly moving averages:

Year	Quarter	Values	4-quarterly moving total	Four quarterly moving average	Four quarterly moving average centred
2010	I	70			
	II	60			
			247	$\frac{247}{7} = 61.75$	
	III	45			$\frac{125.75}{2} = 62.875$
			256	$\frac{256}{4} = 64.00$	
	IV	72			$\frac{127}{7} = 63.500$
			252	$\frac{256}{2} = 63.00$	
2011	I	79			$\frac{126.25}{2} = 63.500$
			253	$\frac{253}{4} = 63.25$	
	II	56			$\frac{129.50}{2} = 64.750$

	III	46	265		$\frac{135.25}{2} = 67.625$
			276	$\frac{276}{4} = 69.00$	
	IV	84			$\frac{140}{2} = 70.000$
			284	$\frac{284}{4} = 71.00$	
2012	I	90			$\frac{141.75}{2} = 70.875$
			283	$\frac{284}{6} = 70.75$	
			281	$\frac{281}{6} = 70.25$	
	III	45			
	IV	82			