2016

Part I

Question: 1

[3x9=10]

i. If $A=\begin{bmatrix}3 & -2\\ 4 & -2\end{bmatrix}$, find x such that $A^2 = xA - 2I$. Hence find A^{-1}.

Answer:

$$A = \begin{bmatrix}3 & -2\\ 4 & -2\end{bmatrix}, \text{ given that } A^2 = xA - 2I$$

$$A^2 = \begin{bmatrix}3 & -2\\ 4 & -2\end{bmatrix}\begin{bmatrix}3 & -2\\ 4 & -2\end{bmatrix} = \begin{bmatrix}9-8 & -6+4\\ 12-8 & -8+4\end{bmatrix}\begin{bmatrix}1 & -2\\ 4 & -4\end{bmatrix}$$

Now $A^2 = xA - 2I$

$$\Rightarrow \begin{bmatrix}1 & -2\\ 4 & -4\end{bmatrix} = x\begin{bmatrix}3 & -2\\ 4 & -2\end{bmatrix} - 2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}1 & -2\\ 4 & -4\end{bmatrix} = \begin{bmatrix}3x & -2x\\ 4x & -2x\end{bmatrix} - \begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}$$

$\Rightarrow 3x - 2 - 1 \Rightarrow 3x = 3 \text{ or } x = 1$

Now, $A^2 = xA - 2I$

$\Rightarrow A^2 = 1.A - 2I$

$\Rightarrow A^2.A^{-1} = AA^{-1} - 2I.A^{-1}$

$\Rightarrow A(AA^{-1}) = I - 2A^{-1}$

$\Rightarrow AI = I - 2A^{-1}$

$\Rightarrow 2A^{-1} = 1\text{-}AI$

$$= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}3 & -2\\ 4 & -2\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$2A^{-1}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}3 & -2\\ 4 & -2\end{bmatrix} = \begin{bmatrix}1-3 & 0+2\\ 0-4 & 1+2\end{bmatrix} = \begin{bmatrix}-2 & 2\\ -4 & 3\end{bmatrix}$$

$$A^{-1} = \frac{1}{2} = \begin{bmatrix}-2 & 2\\ -4 & 3\end{bmatrix} \text{ or } \begin{bmatrix}-1 & 1\\ -2 & \frac{3}{2}\end{bmatrix}$$

ii. Find the values of k, if the equation $8x^2 - 16xy + ky^2 - 22x + 34y = 12$ represents an ellipse.

Answer:

$8x^2 - 16xy + ky^2 - 22x + 34y - 12 = 0$

Comparing it with

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

$\Rightarrow a = 8, b = k, h = -8$

For the given equation to represent an ellipse

h2 < ab

$\Rightarrow 64 < 8k$

Or $8k > 64$

$\Rightarrow k > 8$

iii. Solve for x: $\sin(2\tan^{-1}x) = 1$

Answer:

$\sin(2\tan^{-1} x) = 1$

$$\Rightarrow \sin\left[\sin -1\left(\frac{2x}{1+x^2}\right)\right] = 1 \left[\text{As } 2\tan -1x = \sin^{-1}\left(\frac{2x}{1+x^2}\right)\right]$$

$$\Rightarrow \frac{2x}{1+x^2} = 1 \Rightarrow 2x = 1+x^2$$

or $x^2 - 2x + 1 = 0$

$\Rightarrow (x-1)^2 = 0$

$x = 1,1$

iv. Two regression lines are represented by $2x + 3y - 10 = 0$ and $4x + y - 5 = 0$

Answer:

Let the line of regression of y on x is

$2x + 3y - 10 = 0$

$\Rightarrow 3y = -2x + 10$

or $y = \frac{-2}{3}x + \frac{10}{3}$

$\Rightarrow b_{yx} = -2/3$

Then the line of regression of x on y is

$4x + y - 5 = 0$

$4x = -y + 5 = 0$

$\Rightarrow 4x = -y + 5$ or $x = \frac{1}{4}y + \frac{5}{4}$

$\Rightarrow b_{xy} = \frac{1}{4}$

Now, for such cases

$b_{yx}.\, b_{xy} < 1$

$$\Rightarrow b_{yx}.\, b_{xy} = \left(\frac{-2}{3}\right)\left(-\frac{1}{4}\right) = \frac{1}{6} < 1$$

v. Evaluate : $\int \frac{\cos ex}{\log\tan(\frac{x}{2})}dx$

Answer:

$$\int \frac{\text{cosec } x}{\log\tan\left(\frac{x}{2}\right)}dx$$

Put $\log\tan\left(\frac{x}{2}\right) = t$

$$\frac{1}{\tan\frac{x}{2}} \cdot \frac{1}{2}\sec^2\frac{x}{2}dx = dt$$

$$\frac{1}{2}\frac{\cos x/2}{\sin x/2\,\cos^2 x/2}dx = dt$$

or $$\frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}}dx = dt \qquad \Rightarrow \frac{1}{\sin x}dx = dt$$

$$\int \frac{\operatorname{cosec} x}{\log\tan\left(\frac{x}{2}\right)} = \int \frac{dt}{t}$$

$= \log t + c$

$$= \log\log\tan\frac{x}{2} + c$$

vi. Evaluate : $\lim\limits_{y \to 0} \dfrac{y - \tan^{-1}}{y - \sin y}$

Answer:

$$\lim_{y \to 0}\frac{y-\tan^{-1}y}{y-\sin y} \qquad \left[\frac{0}{0}\text{form}\right]$$

$$= \lim_{y \to 0}\frac{1-\frac{1}{1+y^2}}{1-\cos y} \qquad \left[\frac{0}{0}\text{form}\right]$$

$$= \lim_{y \to 0}\frac{0-1(-1)(1+y^2)^{-2}.2y}{1-\cos y}$$

$$= \lim_{y \to 0}\frac{2y}{(1+y^2)^2.\sin y} = \lim_{y \to 0}\frac{2y}{(1+y^2)^2}\left(\frac{y}{\sin y}\right)$$

$$= \frac{2}{(1+0)^2}.1 = 2$$

vii. Evaluate $\displaystyle\int_0^1 \frac{xe^x}{(1+x)^2}dx$

Answer:

$$\int_0^1 \frac{xe^x}{(1+x)^2}dx = \int_0^1 \frac{(x+1)-1}{(1+x)^2}.e^x\,dx$$

$$= \int_0^1 \frac{(x+1)e^x}{(1+x)^2}dx - \int_0^1 \frac{e^x}{(1+x)^2}dx$$

$$= \int_0^1 \frac{e^x}{1+x}dx - \int_0^1 \frac{e^x}{(1+x)^2}dx$$

$$= \left[e^x.\frac{1}{x+1}\right]_0^1 - \int_0^1\left[-\frac{1}{(x+1)^2}\right]dx - \int_0^1\left[-\frac{e^x}{(x+1)^2}\right]dx$$

$$= \left[\frac{1}{2}e^{1} - 1.e^{0}\right] + \int_0^1 \left[-\frac{1}{(x+1)^{2}}\right].e^{x}\,dx - \int_0^1 \left[-\frac{e^{x}}{(x+1)^{2}}\right].e^{x}\,dx$$

$$= \frac{1}{2}e - 1$$

$$= \frac{e}{2} - 1$$

viii. Find the modules and argument of the complex number $\frac{2+i}{4i+(1+i)^2}$

Answer:

Let $z = \frac{2+i}{4i+(1+i)^2} = \frac{2+i}{4i+(1+i)^2}$

$= \frac{2+i}{6i+1+(-1)} = \frac{2+i}{6i}$

$= \frac{(2+i) \times i}{6i \times i} = \frac{2i+i^2}{6(-1)}$

$= \frac{2i-i}{-6} = \frac{1}{6} - \frac{1}{3}i$

$\therefore |z| \sqrt{\left(\frac{1}{6}\right)^2 + \left(-\frac{1}{3}\right)^2} = \frac{\sqrt{5}}{6}$

Now let $\alpha = \tan-1\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{-\frac{1}{3}}{\frac{1}{6}}\right| = \tan^{-1} 2$

Also as $x = \frac{1}{6} > 0,\ y = -\frac{1}{3} < 0$

$\Rightarrow \quad z = \frac{1}{6} - \frac{1}{3}$ i lies in fourth quadrant

Hence amp (z) = -tan-1 α = $-\tan^{-1} 2$.

ix. A word consists of 9 different alphabets. In which there are 4 consonants and 5 vowels. Three alphabets are chosen at random. What is the probability that more than one vowel will be selected?

Answer:

No. of alphabets in one word = 9
No. of consonants (C) = 4
No. of vowels (V) = 5
Three alphabets are chosen at random
∴ Probability that more than one vowel will be selected
= P(2 V & Ic) + P (3V)

$= \frac{{}^5C_2 \times {}^4C_1}{9C_3} + \frac{{}^5C_2 \times {}^4C_1}{9C_3} \frac{{}^5C_3}{9C_3} = \frac{40}{84} + \frac{10}{84}$

$= \frac{50}{84} = \frac{25}{42}$

x. Solve the differential equation: $\frac{dy}{dx} = ex^{+y} + x^{2}\ e^{y}$ [3]

Answer:

$\frac{dy}{dx} = e^{x+y} + x^{2}\ e^{y}$

$= e^{x}.e^{y} + x^{2}.\ e^{y}$

$\frac{dy}{dx} = e^{y}\ (e^{x} + \ e^{y})$

$\int \frac{dy}{dx} = \int (e^{x} + x^{2})dx$

$\int e - y\ dy, - e^{2}y = \int e^{x} + dx + \int e^{2}dx$

$-e^{-y} = e^{x} + \frac{x^{3}}{3} + c$

$-\frac{1}{e^{y}} = e^{x} + \frac{x^{3}}{3} + c$

or $e^{x} + y + \frac{e^{y}.x^{3}}{3} + ce^{y} + 1 = 0$

Section A (Higher Analysis) *(Question numbers 2 to 7)*

Part II

Question: 2

[5+5=10]

a. By using properties of determinants, show that $p\,a^2 + 2qa + r = 0$, given that:

p,q ane r not in G.P and $\begin{vmatrix} 1 & \frac{q}{p} & a+\frac{q}{p} \\ 1 & \frac{r}{p} & a+\frac{r}{p} \\ p\,a+q & q\,a+r & 0 \end{vmatrix} = 0$

Answer:

$$\Rightarrow \begin{vmatrix} 0 & \frac{q^2-pr}{pq} & \frac{q^2-pr}{pq} \\ 1 & \frac{r}{q} & \alpha+\frac{r}{q} \\ p\alpha+q & q\alpha+r & 0 \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} 0 & 0 & \frac{q^2-pr}{pq} \\ 1 & -\alpha & \alpha+\frac{r}{q} \\ p\alpha+q & q\alpha+r & 0 \end{vmatrix} = 0$$

$$\Rightarrow \frac{q^2-pr}{pq}[(q\alpha+r)+(p\alpha+q)\alpha] = 0$$

$$\Rightarrow (q^2-pr)[q\alpha+r+p\alpha^2+q\alpha] = 0$$

or $(q^2-pr)\,[p\alpha^2+2q\alpha+r] = 0$

$\Rightarrow \frac{p}{q} \neq \frac{q}{r}$ i.e. $pr \neq q^2$

or $q^2 = pr \neq 0$

$\Rightarrow p\alpha^2 + 2q\alpha + r = 0$

xi. Solve the following system of equations using matrix method:

$$\frac{2}{x}+\frac{3}{y}+\frac{10}{z} = 4$$

$$\frac{4}{x}-\frac{6}{y}+\frac{5}{z} = 1$$

$$\frac{6}{x}-\frac{9}{y}+\frac{20}{z}=2$$

Answer:

$$D = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & 20 \end{bmatrix} = 2(120-45)-3(-80-30)+10(36+36) \quad \left[\therefore \textit{expanding w.r.t. } R_1\right]$$

$$= 150 + 330 + 720 = 1200$$

$$D_1 = \begin{bmatrix} 4 & 3 & 10 \\ 1 & -6 & 5 \\ 2 & 9 & 20 \end{bmatrix} = 4(120-45)-3(-20-10)+10(9+12) \left[\therefore \textit{expanding w.r.t. } R_1\right]$$

$$= 300 + 90 + 210 = 600$$

$$D_2 = \begin{bmatrix} 2 & 4 & 10 \\ 4 & 1 & 5 \\ 6 & 2 & -20 \end{bmatrix} = 2(-20-10)-4(-80-30)+10(8-6) \left[\therefore \textit{expanding w.r.t. } R_1\right]$$

$$= 60 + 440 + 20 = 400$$

$$D_3 = \begin{bmatrix} 2 & 3 & 4 \\ 4 & -6 & 1 \\ 6 & 2 & 2 \end{bmatrix} = 2(-12-9)-3(8-6)+4(36-36) \left[\therefore \textit{expanding w.r.t. } R_1\right]$$

$$= -42-6+288 = 240$$

Now $\frac{D_1}{D}=\frac{1}{x}$ $\qquad \Rightarrow \frac{1}{x}=\frac{600}{1200}=\frac{1}{2} \Rightarrow x = 2$

$\frac{D_2}{D}=\frac{1}{y}$ $\qquad \Rightarrow \frac{1}{x}=\frac{400}{1200}=\frac{1}{3} \Rightarrow y = 3$

$\frac{D_3}{D}=\frac{1}{2}$ $\qquad \Rightarrow \frac{1}{x}=\frac{600}{1200}=\frac{1}{2} \Rightarrow z = 5$

Question: 3 [5+5=10]

a. Prove that: $2\tan -1\frac{1}{5}+\cos -1\frac{7}{5\sqrt{2}}+2\tan -1\frac{1}{8}=\frac{\pi}{4}$ [5]

Answer:

$$\text{L.H.S} = 2\tan^{-1}\frac{1}{5}+\text{Cos}^{-1}\frac{7}{5\sqrt{2}}+2\tan^{-1}\frac{1}{8}$$

$$=2\left(\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}\right)+\text{Cos}^{-1}\frac{7}{5\sqrt{2}}$$

$$=2\tan^{-1}\left[\frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}\text{x}\frac{1}{8}}\right]+\cos^{-1}\frac{7}{5\sqrt{2}} \qquad \left[\because \tan^{-1}x+\tan^{-1}y\left(\frac{x+y}{1-xy}\right)\right]$$

$$= 2\tan^{-1}\left[\frac{\frac{13}{40}}{\frac{39}{40}}\right] + \tan^{-1}\left[\frac{\sqrt{1-\frac{40}{50}}}{\frac{7}{7}\sqrt{2}}\right] \left[\because \cos^{-1}x = \tan^{-1}\left(\sqrt{\frac{1-x^2}{x}}\right)\right]$$

$$= 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{1}{3} + \left(\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}\right)$$

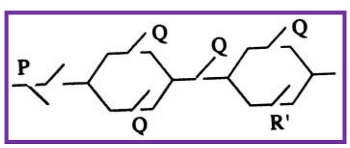
$$= \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{\left(\frac{1}{3}+\frac{1}{7}\right)}{1-\frac{1}{3}\cdot\frac{1}{7}}$$

$$= \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{2}$$

$$= \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{2}}{1-\frac{1}{3}\cdot\frac{1}{2}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{5}{6}}{\frac{5}{6}}\right) = \tan^{-1}1 = \frac{\pi}{4}$$

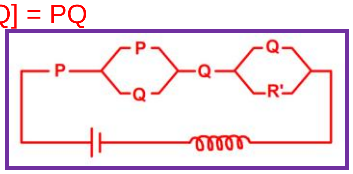
b. P,Q and R represents switches in 'ON position' and P1, Q1 and R1 represent switches in 'OFF position.' Construct a switching a circuit representing the polynomial: P(P+Q) Q(Q+R1). Use Boolean Algebra to show that the above circuit is equivalent to a switching circuit in which when P and Q are in the 'ON position', the light is on

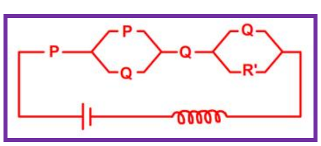


Answer:

We have $P(P+Q)Q(Q+R^1) = [P + PQ]\ [Q + QR^1]$ $\quad$ [$\because$ PP = P, QQ = Q]

$= [P]\ [Q] = PQ$ $\quad$ [$P + PQ = P,\ Q + QR^1 = Q$]

Hence 

is equivalent to 

Question: 4 [5+5=10]

a. Verify Lagrange's mean value theorem for the function f(x) = sin x – sinx in the interval [0, π)

Answer:

i. The function f(x) = sin x – sin 2x is derivable for all values of x, and hence derivable in [0, π].

ii. Also f' (x) = cos x – 2cos 2x

$\Rightarrow$ By Langrange's mean value theorem

We have $f'(c) = \dfrac{f(\pi) - f(0)}{\pi - 0}$

$\Rightarrow \cos c - 2\cos 2c = \dfrac{0-0}{\pi} = 0$

$\Rightarrow \cos c - 2(2\cos 2c - 1) = 0$

$\Rightarrow 4\cos 2c - \cos c - 2 = 0$

$\Rightarrow \cos c = \dfrac{1 \pm \sqrt{1+32}}{8} = \dfrac{\sqrt{1+33}}{8}$

or $\cos c = \dfrac{1 \pm 5.7}{8}$

$\Rightarrow \dfrac{1+5.7}{8}$ or $\cos c = \dfrac{1-5.7}{8}$

$\Rightarrow$ cos c = 0.84 or cos c = -0.59

$\Rightarrow$ c = 33^0 or c = 126.2 (approx..)

Both values of c lies in the internal [0, π]

Hence the Mean value theorem is verified.

b. Find the equation of the hyperbola whose foci are (0,±, 13) and the length of the conjugate axis is 20.

Answer:

Length of conjugate axis = 20

2b = 20

b = 10

Also foci (0, ± ae) = (0, ± 13)

$\Rightarrow$ ae = 13

$b^2 = a^2 (e^2 - 1)$

$100 = a^2e^2 - a^2$

$100 = 169 - a^2$

$a^2 = 169 - 100$

$a^2 = 69.$

Required Hyperbola = $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$

$\Rightarrow \dfrac{y^2}{69} - \dfrac{x^2}{100} = 1$

Question: 5 [5+5=10]

a. Evaluate: $\displaystyle\int \frac{x^2 - 5x - 1}{x^4 + x^2 + 1} dx$.

Answer:

We have $\int \frac{x^2-5x-1}{x^4+x^2+1} = \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{x^2+x+1}$(I)

[$\because$ $x^4 + x^2 + 1 = (x^2 + x + 1)(x^2 - x + 1)$]

$\Rightarrow x^2 - 5x + 1 = Ax^3 - Ax^2 + Ax + Bx^2 - Bx + B + Cx^3 + Dx^2 + Cx^2 + Dx$

$\Rightarrow$ A + C = 0 ....(1) -A + B + D + C = 1 ......(2)

A-B + C + D = -5 ....(3) & B + D = 1

Solving (1), (2), (3) & (4) Simultaneously we get

A = 0 , B = 3, C = 0, D = -2

Replacing values of A,B,C & D in (I) we get

$$\int \frac{x^2-5x-1}{x^4+x^2+1}dx = \int \frac{3}{x^4+x^2+1}dx - \int \frac{3}{x^2+x+1}$$

$$= 3\int \frac{3}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} - 2\int \frac{dx}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

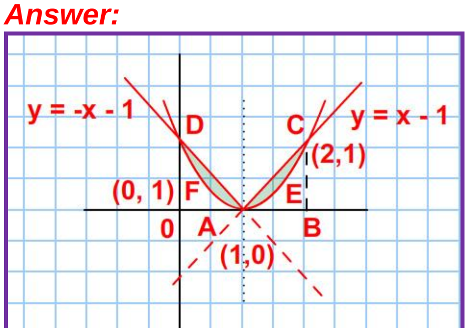
$$3\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\frac{\left(x+\frac{1}{2}\right)}{\frac{\sqrt{3}}{2}} - 2\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\frac{\left(x-\frac{1}{2}\right)}{\frac{\sqrt{3}}{2}} + C$$

$$= \frac{3\times 2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) - \frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

$$= 2\sqrt{3}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) - \frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

b. Draw a rough sketch of the curve y = (x-1)2 and $y = |x-1|$. Find the area of the region bounded by the curve and the x-axis.

Answer:



$y = (x-1)^2$ is a parabola whose axis is a line X = 1 and vertex is at (1,0).

$Y = (X-I)^2$

x	0	1	2
y	1	0	1

Also

$$y = |x-1| = \begin{cases} x-1 \text{ if } x-1 \geq 0 \\ -(x-1) \text{ if } x-1 < 0 \end{cases}$$

$$= \begin{cases} x-1 \text{ if } x-1 \geq 1 \\ -(x-1) \text{ if } x-1 < 1 \end{cases}$$

y = x-1

x	0	1	2
y	-1	0	1

also y = -(x-1)

x	1	1
y	0	0

Here we have to find the area enclosed by y = (x-1)2
And the Lines y = (x-1) and y = -(x-1)
∴ Required Area = Area ABC – Area ABCEA + Area AOD – Area AODFA

$$= \int_1^2 (\text{Line } y = x-1)dx - \int_1^2 [\text{Curve } y = (x-1)^2]dx$$

$$+ \int_1^2 (\text{Line } y = -(x-1))dx - \int_0^1 [\text{Curve } y = (x-1)^2]dx$$

$$= \int_1^2 (x-1)dx - \int_1^2 (x-1)^2 dx + \int_0^1 -(x-1)dx - \int_0^1 (x-1)^2 dx \qquad \lim_{x \to \infty}$$

$$= \left[\frac{x^2}{2} - x\right]_1^2 - \left[\frac{x^3}{2} - x^2 + x\right]_1^2 - \left[\frac{x^2}{2} - x\right]_0^1 - \left[\frac{x^3}{2} - x^2 + x\right]_1^2$$

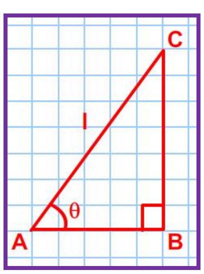
$$= 0 - \left(-\frac{1}{2}\right) - \left[\frac{8}{3} - 4 + 2 - \frac{1}{3} + 1 - 1\right] - \left[\frac{1}{2} - 1\right] - \left[\frac{1}{3} - 1 + 1\right]$$

$$= \frac{1}{2} - \frac{1}{3} + \frac{1}{2} - \frac{1}{3} = 1 - \frac{2}{3} \text{ Sq. units}$$

Question: 6 [5+5=10]

a. If the sum of the lengths of the hypotenuse and a side of a right angled triangle is given, show that the area of the triangle is maximum when the angle between them is $\frac{\pi}{3}$

Answer:



Let 'l' be the length of the hypotenuse of the given right angled Δ ABC at B and $\angle$CBA = θ (in radian measure) & $0 < \theta < \pi/2$
Then, a = AB = l cos θ & BC = l sin θ
Given that, the sum of hypotenuse & a side
Let S = l + a
= l + cosθ = 1 (l + cos θ)
or $l = \frac{S}{\cos\theta}$

now, Area of the Δ ABC = $\frac{1}{2}$. AB.BC

$$= \frac{1}{2}(l\cos\theta).\, l\sin\theta$$

$$= \frac{1}{2} l^2 \sin\theta\cos\theta = \frac{1}{4} l^2 \sin\theta\cos\theta$$

or $A = \frac{\sin^2\theta}{4}\left[\frac{S^2}{(1+\cos\theta)^2}\right] \qquad \left[l = \frac{S}{1+\cos\theta}\right]$

now $\frac{dA}{d\theta} = \frac{S^2}{4}$

$$\frac{\left[(1+\cos\theta)^2\frac{d}{d\theta}\sin 2\theta - \sin\theta\frac{d}{d\theta}(1+\cos\theta)^2\right]}{(1+\cos\theta)^4}$$

Now $\frac{dA}{d\theta} = 0$

$\Rightarrow (1+\cos\theta)^2 .2\cos 2\theta + \sin 2\theta (1+\cos\theta) \sin\theta = 0$

$\Rightarrow 2(1+\cos\theta)[(1+\cos\theta).\cos 2\theta + 2\sin\theta\cos\theta.\ \sin\theta = 0$

or $(1+\cos\theta)\,[1+\cos\theta)(1-2\sin^2\theta)+2\sin^2\theta\cos\theta] = 0$

$(1+\cos\theta)\,[1-2\sin^2\theta+\cos\theta-2\sin^2\theta\cos\theta+2\sin^2\theta\cos\theta] = 0$

$(1+\cos\theta)[1-2(1-\cos^2\theta)^2+\cos\theta = 0$

$(1+\cos\theta)[2\cos^2\theta+2\cos\theta-\cos\theta-1]=0$

$(1+\cos\theta)[2\cos\theta(\cos\theta+1)(2\cos\theta-1) = 0$

$\cos\theta=-1$ or $\cos\theta = \frac{1}{2} = \cos 60^0$

$\theta = 60^0$ i.e. $\theta = \frac{\pi}{3}$

b. If y = xx, prove that: $\frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0$

Answer:
Given $y = x^x$
taking log on both sides, we get
log y = x log x
differentiating with reference to x

$$\frac{1}{y}\frac{dy}{dx} = x.\frac{1}{x} + \log x$$

$$\frac{dy}{dx} = y(1+\log x)$$

Differentiating again with reference to x

$$\frac{d^2y}{dx^2} = y\left(0+\frac{1}{x}\right) + (1+\log x)\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{y}{x} + \frac{1}{y}\frac{dy}{dx}\cdot\frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0$$

Question: 7 [5+5=10]

a. The following observations are given :

(1,4),(2,8), (3,2), (5,10), (7,16),(8,6), (9,18). Estimate the value of y when the value of x is 10 and also estimate the value of x when the value of y = 5.

Answer:

Here we have

x	x^2	Y	y^2	Xy
1	1	4	16	4
2	4	8	64	16
3	9	2	4	6
4	16	12	144	48
5	25	10	100	50
6	36	14	196	84
7	49	16	256	112
8	64	6	36	48
9	81	18	324	162
45	285	90	1140	530

$$\bar{x} = \frac{45}{9} = 5,\ \bar{y} = \frac{90}{9} = 10$$

$$b_{yx} = \frac{\sum xy - \frac{1}{n}\left(\sum x \sum y\right)}{\sum xy^2 - \frac{1}{n}\left(\sum x^2\right)} = \frac{530 - \frac{1}{9} \text{ x } 45 \text{ x } 90}{285 - \frac{1}{9} \text{ x } 45 \text{ x} 45} = \frac{80}{60} = \frac{4}{3}$$

$$b_{yx} = \frac{\sum xy - \frac{1}{n}\left(\sum x \sum y\right)}{\sum xy^2 - \frac{1}{n}\left(\sum x^2\right)} = \frac{530 - \frac{1}{9} \text{ x } 45 \text{ x } 90}{1140 - \frac{1}{9} \text{ x } 90 \text{ x} 90} = \frac{80}{240} = \frac{1}{3}$$

$\therefore$

$$y - \bar{y} = b_{yx}\,(x - \bar{x})$$

$$y - 10 = \frac{4}{3}(x - 5)$$

$3y - 10 = 4x - 20 \Rightarrow 3y = 4x - 10$

Given x = 10, we get $3y = 4 \times 10 - 10 = 30$

$y = 10$

Again regression line of x on y.

$$x - \bar{x} = b_{yx}(y - \bar{y})$$

$$x - 5 = \frac{1}{3}(y - 10)$$

or $3x - 15 = y - 10$

$3x = y + 5$

Given y = 5, we get $3x = 5 + 5$

$$x = \frac{10}{3}$$

b. Compute Karl Perarson's Coefficient of Correlation between sales and expenditures of a firm for six months:

Sales (in lakhs of Rs.)	18	20	27	20	21	20	21	29
Expenditures (in lakhs of Rs.)	23	27	28	28	28	28	29	30

Answer:

Sales 'x'	Expenditure	xy	x^2	y^2
18	23	414	324	529
20	27	540	400	729
27	28	756	729	784
20	28	560	400	784
21	29	609	441	841
29	30	870	841	900

$$r(x,y) = \frac{\sum xy - \frac{1}{N}\sum x . \sum y}{\sqrt{\sum x^2 - \frac{1}{N}(\sum x)^2}\sqrt{\sum y^2 - \frac{1}{N}(\sum y)^2}}$$

$$= \frac{3749 - \frac{135 \times 165}{6}}{\sqrt{3135 - \frac{(135)^2}{6}}\sqrt{4567 - \frac{(165)^2}{6}}}$$

$$= \frac{\frac{219}{6}}{\sqrt{\frac{585}{6}}\sqrt{\frac{177}{6}}}$$

$$= \frac{219}{\sqrt{585 \text{ x } 177}} = \frac{219}{\sqrt{103545}} = \frac{219}{314.80} = 0.695$$

= 0.7 approx.

Question: 8 [5+5=10]

a. A purse contains 4 silver and 5 copper coins. A second purse contains 3 silver and 7 copper coins. If a coin is taken out at random from one of the purses, what is the probability that it is a copper coin?

Answer:

Probability of 1st purse = Probability of 2nd purse = $\frac{1}{2}$

Now, probability of taking out copper coin from 1st purse = $\frac{1}{2} \text{x} \frac{5}{9} = \frac{5}{18}$

And probability of taking out copper coin from 2nd purse = $\frac{1}{2} \text{x} \frac{7}{10} = \frac{7}{20}$

Hence the probability of copper coin from the two purses = $\frac{5}{18} + \frac{7}{20} = \frac{50+63}{180} = \frac{113}{180}$

b. Aman and Bhuvan throw a pair of dice alternately. In order to win, they have to get a sum of 8. Find their respective probabilities of winning if Aman starts the game.

Answer:

Success of Aman and Bhuvan is getting a total of 8 in a pair of dice.

∴ favourable cases = { (2,6), (3,5), (4,4), (5,3), (6,2)}]

Let p = Probability of getting a total of 8 and q = Probability of not getting a total of 8

$\therefore$ p = $\frac{5}{36}$ and q $\frac{31}{36}$

If A man starts the game, he can win it in the first throw, 3rd throw, 5th throw, 7th throw and so on.

Now, Probability of Aman winning in the third row.

$$= qqp = \frac{31}{36} \text{x} \frac{31}{36} \text{x} \frac{5}{36} = \left(\frac{31}{36}\right)^2 \text{x} \frac{5}{36}$$

Probability of Aman's Winning in 5th throw

$$= qqqqp = \left(\frac{31}{36}\right)^4 \text{x } \frac{5}{36} \text{ and so on}$$

Since all cases are mutually exclusive therefore the chances of Aman's winning the game first.

= P + qqp + qqqqp…………………∞

= P [1 + q2 + q4 + …………………∞

= P

$$\left[\frac{1}{1-q^2}\right] \qquad \left[\text{In aG.P.S}_\infty = \frac{a}{1-r}\right]$$

$$\frac{5}{36}\left[\frac{1}{1-\left(\frac{31}{36}\right)2}\right]$$

$$\frac{5}{36}\left[\frac{36 \text{ x } 36}{36 \text{ x } 36 - 31 \text{ x } 31}\right] = \frac{5}{36} \text{ x } \frac{36 \text{ x } 36}{(5\text{x}67)}$$

$$\frac{5 \text{ x } 36}{5 \text{ x } 67} = \frac{36}{67}$$

Question: 9 [5+5=10]

a. Using De Moivre's theorem, find the value of: $\left(1+i\sqrt{3}\right)^6 + \left(1-i\sqrt{3}\right)^6$

Answer:

Let $I + I\ \sqrt{3} = r\,(\cos\theta + i\sin\theta)$

Then $I = r\cos\theta$

$\sqrt{3} = r\sin\theta$

$\Rightarrow r^2 = 3 + 1 = 4 \ \Rightarrow r = 2$

and $\tan\theta = \dfrac{1}{\sqrt{3}} \ \Rightarrow \theta = \pi/3$

$$\therefore I + i\sqrt{3} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$\text{and } \ I - i\sqrt{3} = 2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$$

$$\text{Now, } (I + i\sqrt{3})^6 + (I - i\sqrt{3})^6 = \left[2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]6 + \left[2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)\right]$$

$$= 2^6\left[\frac{6\pi}{3} + \ i\sin\frac{6\pi}{3}\right] + 2^6\,.\left[\cos\frac{6\pi}{3} - \ i\sin\frac{6\pi}{3}\right]$$

b. $y - x\dfrac{dy}{dx} = x + y\dfrac{dy}{dx}$ when y = 0 and x = 1

Answer:

$y - x\dfrac{dy}{dx} = x + y\dfrac{dy}{dx}$, when y = 0 & x = 1

$y\text{-}x = (x\text{+}y)\dfrac{dy}{dx}$

or $\dfrac{dy}{dx} = \dfrac{y - x}{y + x}$

Put y = vx $\qquad \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dy}{dx} = \dfrac{dv}{dx}$

$$(1) \quad \Rightarrow v + x\frac{dv}{dx} = \frac{vx - x}{vx + x}$$

$$= \frac{x(v - 1)}{x(v + 1)}$$

or $v+x\dfrac{dv}{dx} = \dfrac{v-1}{v+1}$

$x\dfrac{dv}{dx} = x\dfrac{v-1}{v+1} - \dfrac{v}{l}$

$x\dfrac{dv}{dx} = \dfrac{v-1-v^2}{v+1} - \dfrac{-(v^2+1)}{v+l}$

$\dfrac{v+l}{v^2+1}dv = \dfrac{dx}{x}$

or integrating both sides

$\int\dfrac{v+1}{v^2+1}dv + \int\dfrac{dx}{x} = c$

or $\dfrac{1}{2}\int\dfrac{2v}{v^2+1}dv + \int\dfrac{1}{v^2+1}dv + \int\dfrac{dx}{x} = c$

$\Rightarrow \dfrac{1}{2}\log(v^2+1)+\tan^{-1} + \log|x| = c$

$\Rightarrow \dfrac{1}{2}\log\left(\dfrac{y^2+1}{x^2+1}\right)+\tan^{-1} + \log|x| = c$ [replacing $v = \dfrac{y}{x}$]

Now when y = 0 & x = 1

$c = \dfrac{1}{2}\log 1 + \tan^{-1} 0 + \log 1$

C = 0 [$\because \log l = 0$]

$\therefore$ Required solution

is $\dfrac{1}{2}\log\left(\dfrac{y^2+1}{x^2+1}\right)+\tan^{-1}\left(\dfrac{y}{x}\right) + \log|x| = 0$

or $\log\left(\dfrac{x^2+y^2}{x^2}\right)+\tan^{-1}\left(\dfrac{y}{x}\right)+2\log x = 0$

or $\log\left(\dfrac{x^2+y^2}{x^2}\right)+\log x^2 + 2\tan^{-1}\left(\dfrac{y}{x}\right)$

or $\log\left(\dfrac{x^2+y^2}{x^2}\right).x^2 + 2\tan^{-1}$

Section B (Higher Analysis) *(Question numbers 2 to 7)*

Question: 10 [5+5=10]

a. Prove that: $\left[\vec{a}+\vec{b}\;\;\vec{b}+\vec{c}\;\;\vec{c}+\vec{a}\right]=2\left[\vec{a}\;\vec{b}\;\vec{c}\right]$

Answer:

We have

L.H.S $\left[\vec{a}+\vec{b}.\,\vec{b}+\vec{c}\,.\vec{c}+\vec{a}\right]$

$=(\vec{a}+\vec{b}).\left[(\vec{b}+\vec{c})\text{x}(\vec{c}+\vec{a})\right]$

$=(a+b).\left[(\vec{b}\text{ x }\vec{c})+(\vec{b}\text{ x }\vec{a})+(\vec{c}\text{ x }\vec{c})+\vec{c}\text{ x }\vec{a}\right]$

$=(\vec{a}+\vec{b}).\left[(\vec{b}\text{ x }\vec{c})-(\vec{a}\text{ x}\vec{b})+(\vec{c}\text{ x }\vec{a})\right]\left[\text{Q}\vec{c}\text{ x }\vec{c}=0\text{ and }\vec{b}\text{ x }\vec{a}=\vec{a}\text{ x}\vec{b}\right]$

$=\vec{a}.(\vec{b}\text{ x }\vec{c})-\vec{a}.(\vec{a}\text{ x}\vec{b})+\vec{a}.(\vec{c}\text{ x }\vec{a})+\vec{b}.(\vec{b}\text{ x }\vec{c})-\vec{b}.(\vec{b}\text{ x }\vec{a})+\vec{b}.(\vec{c}\text{ x }\vec{a})$

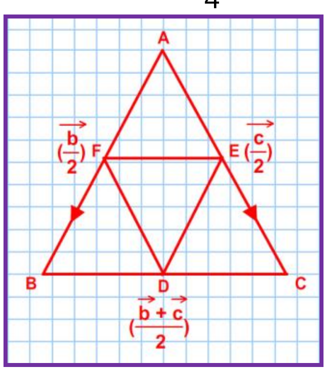
$=\left[\vec{a}\,\vec{b}\,\vec{c}\right]+\left[\vec{b}\,\vec{c}\,\vec{a}\right]$

$=\left[\vec{a}\,\vec{b}\,\vec{c}\right]+\left[\vec{a}\,\vec{b}\,\vec{c}\right]$

$=2\left[\vec{a}\,\vec{b}\,\vec{c}\right]$

b. If D,E, F are mid-points of the sides of a triangle ABC, prove by vector method that:

Area of ΔDEF = $\frac{1}{4}$ (Area of ΔABC)



Answer:

We consider A as origin then the position vectors of B and C may be taken as $\vec{b}$ and $\vec{c}$

$\therefore$ Position Vector of Mid point D = $\dfrac{\vec{b}+\vec{c}}{2}$

Position Vector of E = $\dfrac{\vec{c}}{2}$

Position vector of F = $\dfrac{\vec{b}}{2}$

Hence $\vec{DE}$ = Position Vector of E

And $\vec{DF}$ = Position Vector of F – Position Vector of D

$= \frac{\vec{c}}{2} - \frac{\vec{b}+\vec{c}}{2} = -\frac{\vec{b}}{2}$

Now Vector Area of Δ DEF = $\frac{1}{2}(\vec{DE} \times \vec{DF})$

$\frac{1}{2}\left(\frac{-\vec{b}}{2}\right) \times \left(\frac{-\vec{c}}{2}\right)$

$\frac{1}{8}\vec{b} \times \vec{c}$

$\frac{1}{4}\left(\frac{1}{2}\vec{b} \times \vec{c}\right)$

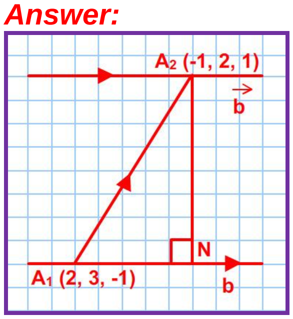
$\frac{1}{4}$ [Area of Δ ABC]

Area of Δ DEF = $\frac{1}{4}$ (Area of Δ ABC)

Question: 11 [5+5=10]

a. Find the vector equation of the line passing through the point (-1,2,1) and parallel to the line $\vec{r} = 2\hat{i} + 3\hat{j} - k + \lambda(\hat{i} - 2\hat{j} + k)$. Also, find the distance between these lines.

Answer:



The given line is $\vec{r} = 2\hat{i} + 3\hat{j} - k + \lambda(\hat{i} - \hat{i})$ ....(i)

It passes through the point A1 with position vector

$\vec{a_1} = 2\hat{i} + 3\hat{j} - k$ and is parallel to the vector

$\vec{b} = \hat{i} - 2\hat{j} - k$

The Equation of the line passing through the point A2 (-1, 2, 1) with position vector

$\vec{a_2} = -\hat{i} + 2\hat{j} + k$ and parallel to the line (i) is

$\vec{r} = \hat{i} + 2\hat{j} + k + \mu(\hat{i} - 2\hat{j} + k)$

Also $\vec{a_2} - \vec{a_1} = -3\hat{i} - \hat{j} - 2k$

$|\vec{b}| = \sqrt{(1)^2 + (2)^2 + (1)^2} = \sqrt{6}$

and $\vec{b} \times (\vec{a_2} - \vec{a_1}) = \begin{vmatrix} \hat{i} & \hat{j} & k \\ 1 & -2 & 1 \\ -3 & -1 & 2 \end{vmatrix}$

$|\vec{b} \times (\vec{a_2} - \vec{a_1})| = \sqrt{(-3)^2 + (-5)^2 + (-7)^2}$

$= \sqrt{9+25+49} = \sqrt{83}$

$\therefore$ The distance between the parallel lines (i) & (ii)

$$\frac{\left|\vec{b} \times (\overrightarrow{a_2} - \overrightarrow{a_1})\right|}{|b|}$$

$$\sqrt{\frac{83}{6}}$$

$$\sqrt{\frac{83}{6}} \text{ units}$$

b. Find the equation of the plane passing through the points A (2,1, -3), B(-3,-2,1) and C(2,4,-1). [5]

Answer:

Equation of the plane passing through A (2, 1 -3) is

A(x-2) + b(y-1) + c (z+3) = 0

Since it passes through B (-3, -2, 1) and C (2, 4, -1) ……(1)

we get

a(-3-2) + b(-2-1) + c (1+3) = 0

i.e. -5a – 3b + 4c = 0 $\Rightarrow$ 5a + 3b – 4c = 0 ……(2)

and a (2-2) + b(4-1) + c(-1+3) = 0

i.e. 3b+ 2c = 0 …….(3)

solving (2) & (3)

a:b:c = 18:-10:15

replacing in (1) required equation is

18(x-2) – 10 (y-1) + 15 (z+3) = 0

18x – 36 – 10y + 10 +15z + 45 = 0

18x – 10y + 15z + 19 = 0

Question: 12 [5+5=10]

a. A box contains 4 red and 5 black marbles. Find the probability distribution of the red marbles in a random draw of three marbles. Also find the mean and standard deviation of the distribution.

Answer:

Let X denotes the number of Red marbles drawn from the box. Since there are 4 Red marbles therefore X can take the values 0,1,2 & 3.

P(X=0) = Probability of getting no red marbles

= Probability of 3 marbles drawn are Black

$$= \frac{{}^5C_3}{{}^9C_3} = \frac{10}{84} = \frac{5}{42}$$

P(X-1) = Probability of getting 1 Red marbles

$$\frac{{}^4C_1 \times {}^5C_2}{{}^9C_3} = \frac{4 \times 10}{84} = \frac{10}{21}$$

P(X-2) = Probability of getting 2 Red marbles

$$\frac{{}^4C_2 \times {}^5C_1}{{}^9C_3} = \frac{6 \times 5}{84} = \frac{5}{14}$$

P(X-3) = Probability of getting 3 Red marbles

$$\frac{{}^{4}C_{3}}{{}^{9}C_{3}} = \frac{4}{84} = \frac{1}{21}$$

$\therefore$ The Probability distribution of X is given as:

X	0	1	2	3
P(X)	$\frac{5}{42}$	$\frac{10}{21}$	$\frac{5}{14}$	$\frac{1}{21}$

Again

X_i	$P_i(x)$	$P_i \times X_i$	X_i^2	$P_i \times X_i^2$
0	$\frac{5}{42}$	0	0	0
1	$\frac{10}{21}$	$\frac{10}{21}$	1	$\frac{10}{21}$
2	$\frac{5}{14}$	$\frac{5}{7}$	4	$\frac{10}{7}$
3	$\frac{1}{21}$	$\frac{1}{7}$	9	$\frac{3}{7}$
		$\sum P_iX_i = \frac{4}{3}$		$\sum P_iX_i \times X_i^2 = \frac{7}{3}$

Mean = $\mu = \sum P_iX_i = \frac{4}{3}$

$$\sigma = \sqrt{\Sigma P_iX_i^2 - \mu^2}$$

$$= \sqrt{\frac{7}{3} - \frac{16}{9}} = \sqrt{\frac{21-16}{9}} = \sqrt{\frac{5}{3}}$$

b. Bag A contains 2 white, 1 black and 3 red balls, Bag B contains 3 white, 2 black and 4 red balls and Bag C contains 4 white, 3 black and 2 red balls. One Bag is chosen at random and 2 balls are drawn at random from that Bag. If the randomly drawn balls happen to be red and black, what is the probability that both balls come from Bag B?

Answer:

Bag A	Bag B	Bag C
2 W	3 W	4 W
1 B	2 B	3 B
3 R	4 R	2 R

Let E_1, E_2 and E_3 be the events of choosing Bag A, Bag B and Bag C respectively,

$\therefore \quad P(E_1) = p(E_2) = P(E_3) = \frac{1}{3}$

Let E be the event of drawing 1 Red and 1 Black balls from the bags then $P(\frac{E}{E_1})$ = Probability that 1 Red and 1 Black balls are drawn from Bag A.

$$= \frac{{}^{3}C_1 \text{ x } {}^{1}C_1}{{}^{6}C_2} = \frac{3}{15} = \frac{1}{5}$$

Similarly $\quad P(\frac{E}{E_2}) = \frac{{}^{4}C_1 \text{ x } {}^{2}C_1}{{}^{6}C_2} = \frac{4\text{x}2}{36} = \frac{2}{9}$

and $\quad P(\frac{E}{E_2}) = \frac{{}^{2}C_1 \text{ x } {}^{3}C_1}{{}^{6}C_2} = \frac{2\text{x}3}{36} = \frac{1}{6}$

$$\frac{P(E_3).P(\frac{E}{E3})}{P(E_1).P(\frac{E}{E_1}) + P(E_2).P(\frac{E}{E_2}) + P(E_3).P(\frac{E}{E3})}$$

$$\frac{\frac{1}{3} \text{X} \frac{1}{6}}{\frac{1}{3} \text{X} \frac{1}{5} + \frac{1}{3} \text{X} \frac{2}{9} + \frac{1}{3} \text{X} \frac{1}{6}} = \frac{\frac{1}{18}}{\frac{18+20+15}{270}} = \frac{270}{18 \text{ x} 53} = \frac{15}{53}$$

Section C (Statistics) *(Question numbers 13 to 15)*

Question 13 [5+5=10]

a. The price of a tape recorder is Rs. 1661. A person purchases it by making a cash payment of Rs 400 and agrees to pay the balance with due interest in 3 half-yearly equal installments. If the dealer charged interest at the rate of 10% per annum compounded half-yearly, find the value of the installment.

Answer:

Let each installment is Rs. X.

$\therefore$ Total Price of Tape recorder = Rs. 1661

Cash paid = Rs. 400

Remaining Price = 1261

Rate of interest = 10% p.a = 5% (half-yearly)

Now, using the formula for three equal instalments

$$1261 = x\left[\frac{100}{100+5}+\left(\frac{100}{100+5}\right)^2+\left(\frac{100}{100+5}\right)^3\right]$$

$$= x\left[\frac{20}{21}+\left(\frac{20}{21}\right)^2+\left(\frac{20}{21}\right)^3\right]$$

$$= x.\frac{20}{21}\left[1+\frac{20}{21}+\frac{400}{441}\right]$$

$$\text{or, } x = \frac{1261x441x21}{20x1261}=\frac{9261}{20} = 463.05$$

$\therefore$ Each instalment = Rs,. 463.05

b. A manufacturer manufactures two types of tea cups. A and B. Three machines are needed for manufacturing the tea cups. The time in minutes required for manufacturing each cup on the machines is given below: [5]

Type of Cup	Times in minutes		
	Machine I	Machine II	Machine III
A	12	18	6
B	6	0	9

Each machine is available for a maximum of six hours per day. If the profit on each cup of type A is Rs. 1.50 and that on each cup of type B is Rs. 1.00, find the number of cups of each type that should be manufactured in a day to get maximum profit.

Answer:

Let x be the no. of A type tea cups and 'y' be the no. of B type tea cups.

The problem can be formulated as

Maximize the Profit P = 1.50 x + 1.00 y

i.e. P = 1.50x + y

Subject to the constraints

$12x + 6y \le 3600$ i.e. $2x + y \le 600$

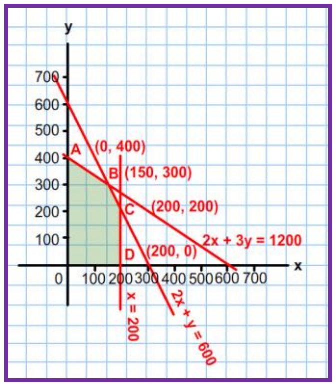
$18x + 0.y = \le 3600$ i.e. $x \le 200$

$6x + 9y \le 3600$ i.e $2x + 3y \le 1200$ and $x \ge 0$, $y \ge 0$

We draw the straight lines. $2x + y = 600$

$x = 200$, $2x + 3y = 1200$, $x = 0$ and $y = 0$

The shaded portion shows the feadible region. The pts. B and C are the points of intersection of lines 2x+ y = 600, with 2x+3y = 1200 and x = 200 with 2x+ y = 600 respectively.
Thus the 5 corner points of the feasible region are 0(0,0), A(0,400) B (150, 300), C (200, 200) and D (200,0)
Clearly the maximum Profit is at x = 150 and y = 300.
Thus the manufacturer should manufacture 150 cups of type A and 300 cups of type B to get maximum profit in a day.



Question 14 [5+5=10]

a. If the difference between Banker's discount and True discount of a bill for 73 days at 5% per annum is Rs. 10, find (i) amount of the bill (ii) the Banker's discount.

Answer:
i. Here we have
n = 73 days

$= \frac{73}{365}$ yrs $= \frac{1}{5}$ yrs.

$i = \frac{5}{100}$

Given that B.D – T.D = 10
We know B.G = B.D – T.D

$$= \frac{A(ni)^2}{1+ni}$$

$$10 = \frac{A\left[\frac{1}{5} \text{ x } \frac{5}{100}\right]^2}{\left(1+\frac{1}{5} \text{ x } \frac{5}{100}\right)}$$

$$= \frac{\frac{A}{10,000}}{\frac{101}{100}}$$

$$\Rightarrow 10 = \frac{A}{10,000}$$

$\Rightarrow$ A = 101000
(ii) We know Banker's Discount (B.D) = Ani

$= 101000 \times \frac{1}{5} \times \frac{1}{100}$

= Rs. 1010

c. Given that the total cost function for x units of a commodity is : $C(x) = \frac{x^3}{3} + 3x^2 - 7x + 16$

(i) Find the Marginal Cost (MC) [5]

Answer:

Marginal cost (M.C) $= \frac{d}{dx}\left[\frac{x^3}{x} + 3x^2 - 7x + 16\right]$

$= x^2 + 6x - 7$

(ii) Find the Average Cost (AC)

Answer:

Average Cost (AC) $= \frac{\text{Total Cost}}{\text{Output}} = \left[\frac{x^3}{x} + 3x^2 - 7x + 16\right] = \frac{x^3}{x} + 3x^2 - 7x + 16$

(iii) Prove that: Marginal Average Cost (MAC) $= \frac{x(MC)\ C(x)}{x^2}$

Answer:

Marginal Average Cost (MAC) $= \frac{d}{dx}\left[\frac{x^3}{x} + 3x - 7x + 16\right] = \frac{2x}{3} + 3 - \frac{16}{x^2}$

Again, $\frac{x(MC) - C(x)}{x^2} = \frac{x(x^2 + 6x - 7) - \frac{x^3}{3} - 3x^2 + 7x - 16}{x^2}$

$= \frac{1}{x^2}\left[x^3 + 6x^2 - 7x - \frac{x^3}{3} - 3x^2 + 7x - 16\right]$

$= \frac{1}{x^2}\left[x^3 - \frac{x^3}{3} - 3x^2 + 7x - 16\right]$

$= \frac{2x^3}{3x^2} + 3 - \frac{16}{x^2}$

$= \frac{2}{3}x + 3 - \frac{16}{x^2}$

$\Rightarrow$ Marginal Average Cost (MAC) $= \frac{x(MC) - C(x)}{x^2}$

Question 15

[5+5=10]

a. The price quotations of four different commodities for 2001 and 2009 are as given below. Calculate the index number of 2009 with 2001 as the base year by using weighted average of price relative method.

Commodity	**Weight**	**Price in (Rs.)**	
		2009	2001
A	10	9.00	4.00
B	49	4.40	5.00
C	36	9.00	5.00
D	4	3.60	2.00

Answer:

Commodity	Weight (w)	Price (in) 2001 (P_0)	Price (in) 2001 (P_1)	Price Relative I = $\frac{P_1}{P_2}$x100	tw
A	10	4.00	9.00	225	2250
B	49	5.00	4.40	88	2250
C	36	6.00	9.00	150	5400
D	4	2.00	3.60	180	720

$\sum w = 99$ $\sum Iw = 12682$

Index Number = $\frac{\sum Iw}{\sum w} = \frac{12682}{99} = 128.101$

b. The profit of a soft drink firm (in thousands of rupees) during each month of the year is as given below:

Months	**Profit (in thousands of Rupees)**
January	3.6
February	4.3
March	4.3
April	3.4
May	4.4
June	5.4
July	3.4
August	2.4
September	3.4
October	1.8
November	0.8
December	1.2

Calculate the four monthly moving averages and plot these and the original data on a graph sheet.

Answer:

Months	Profit (in thousands of Rupees)	4 Monthly moving total	4 Monthly moving average	4 Monthly moving average centred
January	3.6			
February	4.3	15.4		
March	4.3	16.4		
April	3.4	17.5		
May	4.4	16.6		
June	5.4	15.6		
July	3.4	14.6		
August	2.4	11.0		
September	3.4	8.4		
October	1.8	7.2		
November	0.8			
December	1.2