2017

Part I

Question: 1

[3 × 10 = 30]

i. If $A = \begin{bmatrix} 6 & x & 2 \\ -2 & -1 & 2 \\ -10 & 5 & 2 \end{bmatrix}$ is a singular matrix, find the value of x.

Answer:

A is a singular matrix.

$\therefore |A| = 0$

$\Rightarrow 6(-12) - x(24) + 2(0) = 0$

$\Rightarrow -24x = 72$

$x = -3$

ii. Solve: $\text{Cos}^{-1}[\sin(\cos\text{-}1x)] = \frac{\pi}{3}$

Answer:

$\cos^{-1}[\sin(\cos^{-1}x)] = \frac{\pi}{3}$

$\Rightarrow \sin\left(\sin^{-1}\sqrt{1-x^2}\right) = \cos\frac{\pi}{3}$

$\Rightarrow \sqrt{1-x^2} = \frac{1}{2}$

$\Rightarrow 1 - x^2 = \frac{1}{4}$

$\Rightarrow x^2 = \frac{3}{4}$

$x = \pm\frac{\sqrt{3}}{2}$

iii. Show that the line $y = x + \sqrt{7}$ touches the hyperbola $9x^2 - 16y^2 = 144$

Answer:

$y = x + \sqrt{7}$ ….(1)

$9x^2 - 16y^2 = 144$ …………(2)

$m = 1, c = \sqrt{7} \Rightarrow \quad \frac{x^2}{16} - \frac{y^2}{9} = 1$

Touching condition, $\quad \Rightarrow a^2 = 16, b^2 = 9$

$C^2 = a^2m^2 - b^2$

$\Rightarrow 7 = 16 \times 1 - 9$

$\Rightarrow 7 = 16 - 9$

$\Rightarrow 7 = 7$

Hence line $y = x + \sqrt{7}$ touches the hyperbola $9x^2 - 16y^2 = 144$.

iv. Evaluate $\lim\limits_{x\to\frac{\pi}{2}}[x\tan x - \frac{\pi}{2}\sec x]$

Answer:

$$\lim_{x\to\frac{\pi}{2}}\left[x\tan x - \frac{\pi}{2}\sec x\right] = \lim_{x\to\frac{\pi}{2}}\left[\frac{x\sin x - \frac{\pi}{2}}{\cos x}\right]$$

By D.L' Hospital Rule $= \lim\limits_{x\to\frac{\pi}{2}}\left[\frac{x\cos x + \sin x}{-\sin x}\right] = -1$

v. Evaluate : $\int \frac{x}{(x+1)^2}dx$

Answer:

$$\int \frac{x}{(x+1)^2}dx = \int\left[\frac{1}{x+1} - \frac{1}{(x+1)^2}dx\right]$$

$$= \log(x+1) + \frac{1}{x+1} + c$$

vi. Evaluate : $\int_{-3}^{3}|(x+2)|dx$

Answer:

We have $|x+2|$ = x+2, if x $\geq$ -2

= -(x+2) if x < -2

$$\therefore \int_{-3}^{3}|x+2|dx = \int_{-3}^{-2}-(x+2)dx + \int_{-2}^{3}(x+2)dx$$

$$= -\left[\frac{x^2}{2}+2x\right]_{-3}^{-2} + \left[\frac{x^2}{2}+2x\right]_{-2}^{3}$$

$$= -\left[2-4-\frac{9}{2}-2(-3)\right] + \left[\frac{9}{2}+6-2+4\right]$$

$$= -\left[4-\frac{9}{2}\right] + \left[\frac{9}{2}+8\right] = \frac{1}{2}+\frac{9}{2}+8$$

=5 + 8 = 13

vii. A fair die is thrown once. What is the probability that either an even number or a number greater than three will turn up?

Answer:

Required probability = $\frac{4}{6} = \frac{2}{3}$

viii. If the regression equation of x and y is given by $mx - y + 10 = 0$ and the equation of y on x is given by $-2x + 5y = 14$, determine the value of 'm' if the coefficient of correlation between x and y is $\frac{1}{\sqrt{10}}$

Answer:

The regression equation of y on x, $-2x + 5y = 14$

$$\therefore b_{xy} = \frac{1}{m}$$

$$\therefore b_{yx} = \frac{1}{m}$$

$$\therefore b_{yx} = \frac{2}{5}$$

$$\therefore r^2 = b_{xy}.b_{yx}$$

$$\therefore \frac{1}{10} = \frac{2}{5} \text{ x } \frac{1}{m}$$

$$\therefore m = \frac{10 \text{ x } 2}{5} = 4$$

$$\Rightarrow m = 4$$

ix. If 1, ω, ω^2 are the three cube roots of unity, then simplify: $(3+5\omega+3\omega^2)^2\ (1+2\omega+\omega^2)$ [3]

Answer:

$$(3+5\omega+3\omega^2)2(1+2\omega+\omega^2) = [3(1+\omega+\omega^2)+2\omega]^2\ (1+\omega+\omega^2+\omega)$$

$$= (2\omega)^2(\omega) = 4\omega^2.\omega = 4\omega^3$$

$= 4$

x. Solve the differential equation : $\text{cosec}^3 x\ dy - \text{cosec}y\ dx = 0$

Answer:

$\text{cosec}^3\ xdy - \text{cosec}\ y\ dx = 0$

$\Rightarrow \sin y\ dy = \sin^3 x\ dx$

Integrating both sides,

$$\Rightarrow \int \sin y\ dy = \frac{1}{4}\int[3\sin x - \sin 3x]\ dx + c$$

$$\Rightarrow \cos y = \frac{1}{4}\left[3\cos x \frac{\cos 3x}{3}\right] + c$$

Section A (Higher Analysis)

Part II

Question: 2 [4+3=7]

a. By using properties of determinants, prove that the determinant

$$\Delta=\begin{vmatrix} a & \sin x & \cos x \\ -\sin x & -a & 1 \\ \cos x & 1 & a \end{vmatrix}$$ is independent of x.

Answer:

$$\Delta = \begin{vmatrix} a & \sin x & \cos x \\ -\sin x & -a & 1 \\ \cos x & 1 & a \end{vmatrix}$$

$$\Delta = \begin{vmatrix} a+\sin x & \sin x & \cos x \\ -(a+\sin x) & -a & 1 \\ 1+\cos x & 1 & a \end{vmatrix} (\text{By } C_1 \to C_1 + C_2)$$

$$\Delta = \begin{vmatrix} 0 & -a+\sin x & 1+\cos x \\ -(a+\sin x) & -a & 1 \\ 1+\cos x & 1 & a \end{vmatrix} (\text{By } R_1 \to R_1 + R_2)$$

Now expanding,

$\Delta = -(-a + \sin x)\,[-a^2 - a\sin x - 1 - \cos x)]\,[-a - \sin x + a + a\cos x]$

$= -[a^3 + a^2 \sin x + a\cos x - a^2 \sin x - a\sin^2 x - \sin x - \sin x \cos x] + [a\cos x - \sin x - \sin x\cos x + a\cos^2 x]$

$= a^3$, which is independent of x.

b. Using matrix method, solve the following equations:

5x + 3y + z = 16
2x + y + 3z = 19
x + 2y + 4 = 25

Answer:

Writing the given system of equations in matrix form

$$\begin{bmatrix} 5 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ 19 \\ 25 \end{bmatrix} \text{ or } AX = B$$

$|A| = 5(4-6) - 3(8-3) + (4-1)$

$= -10 - 15 + 3 = 22$

$A_{11} = -2$, $A_{12} = -5$, $A_{13} = 3$

$A_{21} = -10$, $A_{22} = 19$, $A_{23} = -7$

$A_{31} = 8$, $A_{32} = -13$, $A_{33} = -1$

$$\therefore \text{adj } A = \begin{bmatrix} -2 & -10 & 8 \\ -5 & 19 & -13 \\ 3 & -7 & -1 \end{bmatrix}$$

$$\therefore\ A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{-22}\begin{bmatrix} -2 & -10 & 8 \\ -5 & 19 & -13 \\ 3 & -7 & -1 \end{bmatrix}$$

$$\therefore\ X = A^{-1}B = \frac{1}{-22}\begin{bmatrix} -2 & -10 & 8 \\ -5 & 19 & -13 \\ 3 & -7 & -1 \end{bmatrix}\begin{bmatrix} 16 \\ 19 \\ 25 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{-22}\begin{bmatrix} -32 & -190 & +200 \\ -80 & +361 & -325 \\ 48 & -133 & -25 \end{bmatrix} = \frac{1}{-22}\begin{bmatrix} -22 \\ -44 \\ -110 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} \Rightarrow x = 1, y = 2, z = 5$$

Question: 3 [5+5=10]

a. Using Rolle's theorem, find a point on the curve y = sin x + cos x – 1, x $\varepsilon\left[0, \frac{\pi}{2}\right]$ where the tangent is parallel to the x-axis.

Answer:

y = f(x) = sin x + cos x – 1

Since cosine and sine function is continuous for all values, so f(x) is continuous in $\left[0, \frac{\pi}{2}\right]$ and derivable in $\left(0, \frac{\pi}{2}\right)$

f(0) = sin 0 + cos 0 – 1 = 0

$f\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} + \cos\frac{\pi}{2} - 1 = 0$

$\Rightarrow f(0) = f\left(\frac{\pi}{2}\right)$

Thus all the three conditions of Rolle's Theorem are satisfied.

$\therefore$ f'(x) = 0 , f'(x) = cos x – sin x

$\Rightarrow$ cos x – sin x = 0

$\Rightarrow \tan x = 1 = \tan\frac{\pi}{4}$

$\Rightarrow x = \frac{\pi}{4} \in \left[0, \frac{\pi}{2}\right]$

$x = \frac{\pi}{4},\ y = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} - 1 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1 = \sqrt{2} - 1$

Hence the required point $\left(\frac{\pi}{4}, \sqrt{2} - 1\right)$

b. Find the equation of the parabola whose focus is (-1,-2) and the equation of the directrix is given by 4x – 3y + 2 = 0. Also find the equation of the axis.

Answer:
According to definition of parabola,
PS = PM

$$(x+1)^2 = (y+2)^2 = \left[\frac{4x - 3(y) + 2}{\sqrt{4^2 + 3^2}}\right]^2$$

$$\Rightarrow x^2 + 2x + 1 + y^2 + 4y + 4 = \frac{16x^2 + 9y^2 + 4 - 24xy - 12y + 16x}{25}$$

$9x^2 + 16y^2 + 24xy + 34x + 112y + 121 = 0$
Equation of axis $3x + 4y = 0$

Question: 4 [5+5=10]

a. If sin-1x + sin-1y + sin-1z = π prove that : $x^2 - y^2 - z^2 + 2yz\sqrt{1-x^2} = 0$

Prove that: $\sin\left[2\tan^{-1}\frac{3}{5} - \text{Sin}^{-t}\frac{7}{25}\right] = \frac{304}{425}$

Answer:

$$\text{L.H.S} = \sin\left[2\tan^{-1}\frac{3}{5} - \text{Sin}^{-t}\frac{7}{25}\right]$$

$$= \sin\left[\tan^{-1}\frac{15}{8} - \text{Sin}^{-t}\frac{7}{25}\right]$$

$$= \sin\left[\sin^{-1}\frac{15}{17} - \text{Sin}^{-t}\frac{7}{25}\right]$$

$$= \sin\left[\sin^{-1}\left(\frac{15}{17}\sqrt{1-\frac{49}{625}} - \frac{7}{25}\sqrt{1-\frac{225}{289}}\right)\right]$$

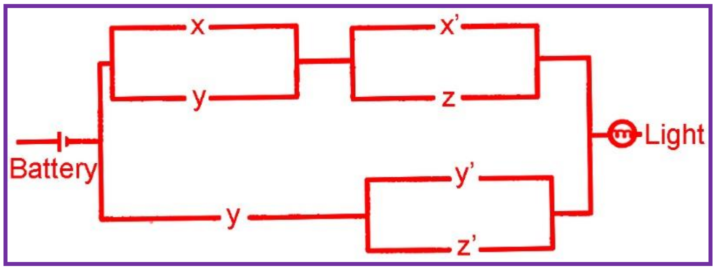
$$= \frac{15}{17} \times \frac{24}{25} - \frac{7}{25} \times \frac{8}{17}$$

$$= \frac{360 - 56}{425} = \frac{304}{425}$$

b. x, y and z represent three switches in an ‘ON’ position and x’, y’ and z’ represent the same three switches in an ‘OFF’ position. Construct a switching circuit representing the polynomial (x+y) (x’+z)+ y(y’+z’)

Using the laws of Boolean Algebra, show that the above polynomial is equivalent to xz + y and construct an equivalent switching circuit.

Answer:
The following is switching circuit.



Now, $(x+y)(x'+z) + y(y' + z') = xx' + xy + yx' + yz + yy' + yz'$

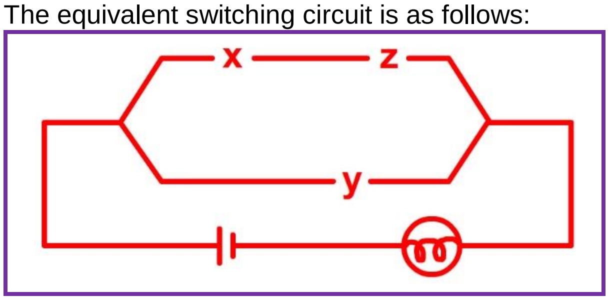
$= xz + yx' + yz + yz'$

$= xz + yx' + y(z+z')$

$= xz + yx' + y$

$= xz + y(x' + 1) = xz + y$

The equivalent switching circuit is as follows:



Question: 5 [5+5=10]

a. Using a suitable substitution, find the derivative of Tan-1 $\sqrt{\frac{a-x}{a+x}}$ with respect to x.

Answer:

$y = \tan^{-1}\sqrt{\frac{a-x}{a+x}}$

Let $x = a\cos 2\theta \Rightarrow \theta \frac{1}{2}\cos^{-1}\frac{x}{a}$

$y = \tan^{-1}\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}} = \tan^{-1}(\tan\theta)$

$\Rightarrow \Rightarrow y = \theta = \frac{1}{2} = \cos^{-1}\frac{x}{a}$

$\frac{dy}{dx} = -\frac{1}{2\sqrt{a^2 - x^2}}$

b. A closed right circular cylinder has volume $\frac{539}{2}$ cubic units. Find the radius and the height of the cylinder so that the total surface area is minimum.

Answer:

Let r be the radius and h the height of the cylinder.

$V = \pi r2h = \frac{539}{2} \Rightarrow h = \frac{539}{2\pi r^2}$

Let S be the total surface area, then

$S = 2\pi rh + 2\pi r^2$

$= 2\pi r\left(\frac{539}{2\pi r^2}\right) + 2\pi r^2$

$S = \frac{539}{2} + 2\pi r^2$

$\therefore \frac{ds}{dr} = -\frac{539}{2} + 2\pi r^2$

For max or min. $\frac{ds}{dr} = 0$

$\Rightarrow \quad - \Rightarrow \frac{539}{2\pi r^2} + 4\pi r = 0$

$\Rightarrow r = \frac{539}{2\pi r^2}^{\frac{1}{3}}$

$\therefore \frac{d^2S}{dr^2} = \frac{1078}{r^3} + 4\pi$ which is positive

$\therefore$ S is minimum, where $r = \left(\frac{4\pi}{539}\right)^{\frac{2}{3}}$

and $\qquad h \frac{539}{2\pi\left(\frac{539}{4\pi}\right)^{\frac{2}{3}}} = \frac{539}{2\pi}\left(\frac{4\pi}{539}\right)^{\frac{2}{3}}$

Question: 6 [5+5=10]

a. Evaluate: $\int \frac{2\sin 2\theta - \cos\theta}{6 - \cos^2\theta - 4\sin\theta} d\theta$.

Answer:

$I = \int \frac{2\sin 2\theta - \cos\theta}{6 - \cos 2\theta - 4\sin\theta} d\theta$

$I = \int \frac{4\sin\theta\cos\theta - \cos\theta}{6 - 1 + \sin^2\theta\cos 2\theta - 4\sin\theta} d\theta$

$= \int \frac{\cos\theta(4\sin\theta - 1)}{\sin^2\theta - 4\sin\theta + 5} d\theta$

Sin θ = t

Let cos θ dθ = dt

$I = \int \frac{4t - 1}{t^2 - 4t + 5} dt$

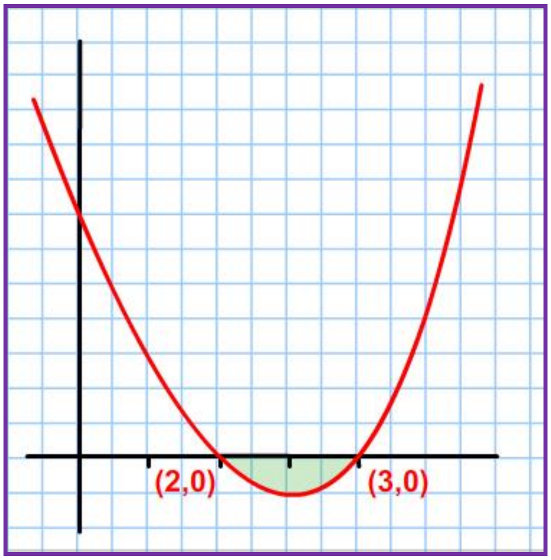
$= 2\int \frac{2t - 4}{t^2 - 4t + 5} dt + 7\int \frac{4t - 1}{t^2 - 4t + 5}$

$= 2 \log (t^2 - 4t + 5) + 7 \tan^{-1} (t-2) + C$

$= 2 \log (\sin^2\theta - 4 \sin\theta + 5) + 7 \tan^{-1} (\sin\theta - 2) + C$

b. Draw a rough sketch of the curve y = x2 – 5x + 6 and find the area bounded by the curve and the x-axis.

Answer:



$y = x^2 – 5x + 6 = (x-3)\ (x-2)$
i. curve does not pass through the origin.
ii. the curve cuts the x-axis at (2,0) and (3,0)
iii. the curve cuts the y-axis at (0,6).

The area bounded by the curve and the x-axis is given by

$$\int_2^3 y\,dx = \int_2^3 (x3 - 5x + 6)dx$$

$$= \left[\frac{x^3}{3} - \frac{5x^2}{2} + 6x\right]_2^3$$

$$= 9 - \frac{45}{2} + 18 - \frac{8}{3} + 10 - 12$$

$$= 25 - \frac{151}{6} = \frac{1}{6} \text{ sq.units}$$

Question: 7 [5+5=10]

a. Find the equations of the two lines of regression for the following observations:

(3,6),(4,5), (5,4), (6,3), (7,2). Find an estimate of y for x = 2.5

Answer:
We have $\sum x_i = 3 + 4 + 5 + 6 + 7 = 25$
$\sum y_i = 6 + 5 + 4 + 3 + 2 = 20$
$\sum x_i^2 = 9 + 16 + 25 + 36 + 49 = 135$\

$\sum y_i^2 = 6^2 + 5^2 + 4^2 + 3^2 + 2^2 = 90$

$\sum xiyi = 3.6 + 4,.5 + 5.4 + 6.3 + 7.2 = 90$

$n = 5, \bar{x} = \frac{\sum x_i}{n} = \frac{25}{5} = , \bar{y} = 4$

$$byx = \frac{\sum x_i y_i - \frac{(\sum x_i)(\sum y_i)}{n}}{\sum x_i^2 - \frac{(\sum x_i)^2}{n}} = \frac{90 - \frac{25 \text{ x } 20}{5}}{135 - \frac{625}{5}} = \frac{90-100}{10} = -1$$

$$bxy = \frac{\sum x_i y_i - \frac{(\sum x_i)(\sum y_i)}{n}}{\sum y_i^2 - \frac{(\sum y_i)^2}{n}} = \frac{90 - \frac{25 \text{ x } 20}{5}}{135 - \frac{400}{5}} = \frac{90-100}{90-80} = -1$$

The regression lines are

$y - \bar{y} = byx (x - \bar{x})$ and $x - \bar{x} = bxy (y - \bar{y})$

$y - 4 = -x + 5$ $\quad$ $x-5 = -y+4$

$x+y = 9$ $\quad$ $x+y = 9$

at $x = 2.5$

$\therefore$ $y = 9 - 2.5 = 6.5$

b. Calculate Spearman's coefficient of rank correlation from the following data and interpret the result:

x	16	19	22	28	25	31	37	40	43	49
y	25	25	27	31	27	33	35	41	45	41

Answer:

s.No	X	Y	Rank (x) R_1	Rank (y) R2	Rank difference $D = R_1 - R_2$	D_2
1	16	25	10	9.5	0.5	0.25
2	19	25	9	9.5	-0.5	0.25
3	22	27	8	7.5	0.5	0.25
4	28	31	6	6	0	0
5	25	27	7	7.5	-0.5	0.25
6	31	33	5	5	0	0
7	37	35	4	4	0	0
8	40	41	3	2.5	0.5	0.25
9	43	45	2	1	1	1
10	49	41	1	2.5	-1.5	2.25

Here t = 2, 2,2

$$\therefore R = 1 - \frac{6\left[\sum D^2 + å\left(\frac{t^3 - t}{12}\right)\right]}{N(N^2-1)}$$

$$= 1 - \frac{6\left[4.5 + \frac{3}{2}\right]}{10(99)} = \frac{990 - 36}{990} = \frac{954}{990} = 0.963$$

Question: 8

[5+5=10]

a. Akhil and Vijay appear for an interview for two vacancies. The probability of Akhil's selection is $\frac{1}{4}$ and Vijay's selection is $\frac{2}{3}$. Find the probability that only one of them will be selected.

Answer:

$P(A) = \frac{1}{4} \therefore P(\overline{A}) = 1 - \frac{1}{4} = \frac{3}{4}$

$P(B) = \frac{2}{3} \quad \therefore P(\overline{B}) = 1 - \frac{1}{4} = \frac{3}{4}$

$\therefore$ Probability that only one of them will be selected

$= P(A).\ P(\overline{B}) + P\ (\overline{A}).\ P(B)$

$= \frac{1}{4} \text{x} \frac{1}{3} \text{x} \frac{3}{4} \text{x} \frac{2}{3} = \frac{1+6}{12} = \frac{7}{12}$

b. There are two bags. One bag contains six green and three red balls. The second bag contains five green and four red balls. One ball is transferred from the first bag to the second bag. Then one ball is drawn from the second bag. Find the probability that it is a red ball.

Answer:

Case – I: When a green ball is transferred, then

$P_1 = \frac{6}{9} = \frac{2}{3}$, $P_2 = \frac{4}{10}\ \frac{2}{5}$

$\therefore$ Probability of both the events = $P_1P_2 = \frac{2}{3}\ x\ \frac{2}{5} = \frac{4}{15}$

Case - II: When a red ball is transferred , then

$P_3 = \frac{3}{9} = \frac{1}{3}$ $P_4 = \frac{5}{10} = \frac{1}{2}$

$\therefore$ Probability of both the events = $P_3P_4 = \frac{1}{3}\ x\ \frac{1}{2} = \frac{1}{6}$

$\therefore$ The required probability = $P_1P_2 + P_3P_4 = \frac{4}{15} + \frac{1}{6} = \frac{8+5}{30} = \frac{13}{30}$

Question: 9

[5+5=10]

a. Solve the differential equation: (y+log x) dx – x dy = 0, given that y = 0, when x = 1.

Answer:

$(y+\log x)\,dx - x\,dy = 0 \Rightarrow \frac{dy}{dx} - \frac{y}{x} = \frac{\log x}{x}$

$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = \frac{\log x}{x}$

$\text{I.F} = e^{-\int \frac{1}{x}dx} = e^{-\log x} = \frac{1}{x}$

$y.\frac{y}{x} = \int \frac{\log x}{x^2}dx$

$\Rightarrow \frac{y}{x} = -\frac{1}{x}\log x - \frac{1}{x} + c$

$\Rightarrow \quad y = -\log x - 1 + cx$

$\Rightarrow \; y = 0 , x = 1$

$0 = -1 + c \Rightarrow \; c = 1$

$\therefore \; = -\log x$

b. Find the locus of a complex number z = x + iy, satisfying the relation $|3z - 4i| \le |3z + 2|$. Illustrate the locus in the Argand plane.

Answer:

$|3z - 4i| \le |3z + 2|$

$\Rightarrow |(3y-4)i| \le |3x + 2 + 3iy|$

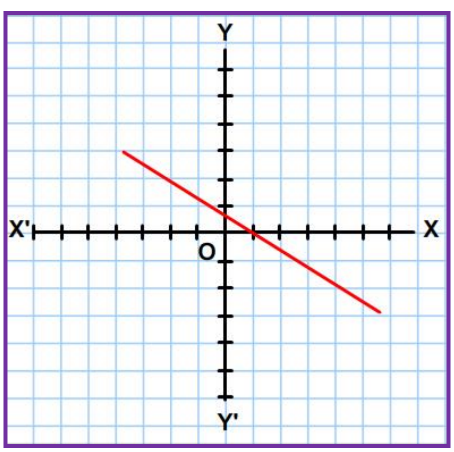
$\Rightarrow \; 9x^2 + (3y - 4)^2 \le (3x+2)^2 + 9y^2$

$\Rightarrow \; 9x^2 + 9y^2 - 24y + 16 \le 9x^2 + 12x + 4 + 9y2$

$\Rightarrow \quad 12x + 24y - 12 \ge 0$

$\Rightarrow \; x+2y - 1 \;\; \ge 0$

x + 2y = 1, which is the required locus of z.
The locus in the Argand plane



Section B *(Question numbers 10 to 12)* [5+5=10]

Question: 10

a. Find the value of λ for which the four points with position vectors $2\hat{i}+5\hat{j}+k$, $-\hat{j}-4k$, $3\hat{i}+\lambda\hat{j}+8k$ and $4\hat{i}+3\hat{j}+8k$ are coplanar.

Answer:

$\vec{a}=-\hat{j}-4k-(2\hat{i}+5\hat{j}+k)$

$=-2\hat{i}-6\hat{j}-5k$

$\vec{b}=(3\hat{i}+\lambda\hat{j}+8k)-(2\hat{i}+5\hat{j}+k)$

$=\hat{i}+(\lambda-5)\hat{j}+7k$

$\vec{c}=4\hat{i}+3\hat{j}+4k-(2\hat{i}+5\hat{j}+k)$

$=-6\hat{i}-2\hat{j}-3k$

$\because \vec{a},\vec{b},\vec{c}$ are coplanar

$\therefore \left[\vec{a},\vec{b},\vec{c}\right]=0$

$$\therefore \begin{vmatrix} -2 & -6 & -5 \\ 1 & \lambda-5 & 7 \\ -6 & -2 & 3 \end{vmatrix}=0$$

$\Rightarrow$ -2(3λ - 15 + 14) + 6 (3+42) – 5 (-2 + 6λ -30) = 0

$\Rightarrow$ -6λ + 2 + 270 + 160 - 30λ = 0

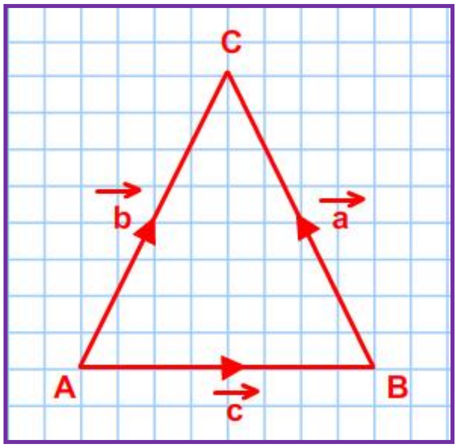
$\Rightarrow$ 36λ = 432

λ = 12

b. In any ΔABC, prove by vector method that $\cos B = \dfrac{c^2+a^2-b^2}{2ca}$

$2\hat{i}+5\hat{j}+k$, $-\hat{j}-4k$, $3\hat{i}+\lambda\hat{j}+8k$ and $4\hat{i}+3\hat{j}+8k$

Answer:



In ΔABC, $\therefore \vec{a}, \vec{b}, \vec{c}$

$\vec{b} = \vec{c} + \vec{a}$

$\vec{b} \cdot \vec{b} = (\vec{c} + \vec{a}) \cdot (\vec{c} + \vec{a})$

$\Rightarrow b^2 = \vec{c} \cdot \vec{c} + \vec{a} \cdot \vec{a} + 2\vec{a} \cdot \vec{c}$

$\Rightarrow b^2 = c^2 + a^2 + 2ac \cos(\pi - B)$

$\Rightarrow 2ac \cos B = c^2 + a^2 - b^2$

$\cos B = \dfrac{c^2 + a^2 - b^2}{2ca}$

Question: 11 [5+5=10]

a. Find the shortest distance between the lines whose vector equations are:
$\vec{r} = (4\hat{i} - \hat{j} + 2k) + \lambda(\hat{i} + 2\hat{j} - 3k)$ and $\vec{r}(2\hat{i} + \hat{j} - k) + \mu(3\hat{i} + 2\hat{j} - 4k)$

Answer:

Shortest distance = $\begin{vmatrix} 4-2 & -1-1 & 2+1 \\ 1 & 2 & -3 \\ 3 & 2 & -4 \end{vmatrix} = \begin{vmatrix} 2 & -2 & 3 \\ 1 & 2 & -3 \\ 3 & 2 & -4 \end{vmatrix}$

= 2 (-8 + 6) + 2 (-4 + 9) + 3 (2-6)
-4 + 10 – 12 = -6
= 6 (Numerically)

b. Find the equation of the plane passing through the line of intersection of the planes x+2y+3z-4 = 0 and 3z – y = 0 and perpendicular to the plane 3x + 4y -2z + 6 = 0

Answer:
Equation of any passing through the intersection of given planes is given by
$(x+2y + 3z - 4) + \lambda (3z - y) = 0$
$\Rightarrow x + (2 - \lambda) - 2 (3 + 3\lambda) = 0$

$\Rightarrow 3 + 8 - 4\lambda - 6 - 6\lambda = 0$
$\Rightarrow -10\lambda = -5$
$\Rightarrow \lambda = \frac{1}{2}$

Hence required equation of plane is given by

$X + (2 - \frac{1}{2})\, y + (3 + \frac{3}{2})\, z - 4 = 0$

$\Rightarrow 2x + 3y + 9z - 8 = 0$
$\Rightarrow 2x + 3y + 9z = 8$

Question: 12 [5+5=10]

a. A factory has three machines A,B and C producing 1,500, 2,500 and 3,000 bulbes per day respectively. Machine A produces 1.5% defective bulbs, machine B produces 2% defective bulbs and machines C produces 2.5% defective bulbs. At the end of the day, a bulb is drawn at random and is found to be defective. What is the probability that this defective bulb has been produced by machine B?

Answer:

Total daily production of bulbs = 1500 + 2500 + 3000

$P(A) = \frac{1500}{7000} = \frac{3}{14}$ $P(B) \frac{2500}{7000} = \frac{5}{14}$. $P(C) = \frac{3000}{7000} = \frac{3}{7}$

$P\left(\frac{F}{A}\right) = 1.5\% = \frac{1.5}{100} = \frac{3}{200}$

$P\left(\frac{F}{B}\right) = 2\% = \frac{2}{100} = \frac{1}{50}$

$P\left(\frac{F}{C}\right) = 2.5\% = \frac{2.5}{100} = \frac{1}{40}$

Now by Baye's theorem, we have

$$\frac{P(B).P\left(\frac{F}{B}\right)}{P(B).P\left(\frac{F}{B}\right) + P\left(\frac{F}{A}\right) + P(C).P\left(\frac{F}{C}\right)}$$

$$= \frac{\frac{5}{14} \text{x} \frac{1}{50}}{\frac{5}{14} \text{x} \frac{1}{50} + \frac{3}{14} \text{x} \frac{3}{200} + \frac{3}{7} \text{x} \frac{1}{40}}$$

$$= \frac{\frac{1}{140}}{\frac{1}{140} + \frac{9}{2800} + \frac{3}{280}} = \frac{20}{59}$$

b. Five bad eggs are mixed with 10 good ones. If three eggs are drawn one by one with replacement, find the probability distribution of the number of eggs drawn.

Answer:

Total eggs = 15

$p=\frac{10}{15}=\frac{2}{3}, q=\frac{1}{3}$ $n=3$

Let X denote the number of good eggs in 3 draws. Then X can take values 0,1,2 and 3.

$\therefore P(X=0)\,{}^3C^0\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^3=\frac{1}{27}$

$P(X=1)={}^3C_1\left(\frac{2}{3}\right)^1\left(\frac{1}{3}\right)^2=3\times\frac{2}{3}\times\frac{1}{9}=\frac{2}{9}$

$P(X=2)$ $P(X=2)={}^3C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^1=3\times\frac{4}{9}\times\frac{1}{3}=\frac{4}{9}$

$P(X=3)={}^3C_3\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^3=\frac{8}{27}$

Hence probability distribution of X is:

X	0	1	2	3
P(X)	$\frac{1}{27}$	$\frac{2}{9}$	$\frac{4}{9}$	$\frac{8}{27}$

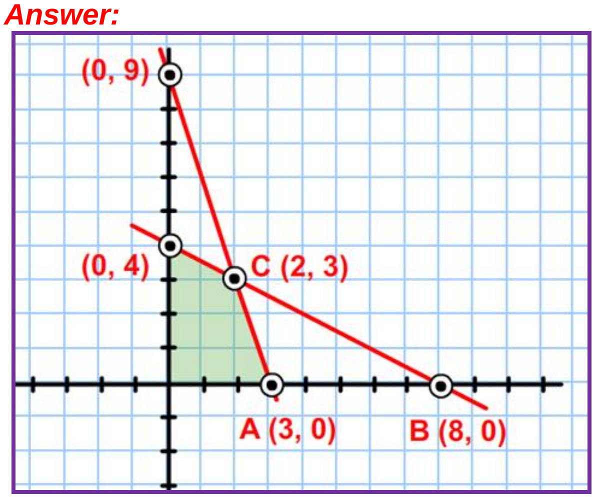
Section C (Statistics) *(Question numbers 13 to 15)*

Question 13

[5+5=10]

a. A company produces two types of items P and Q. manufacturing of both items requires the metals gold and copper. Each unit of item P requires 3gms of copper. The company has 9gms of gold and 8gms of copper in its store. If each unit of item P makes a profit of Rs. 50 and each unit of item Q makes a profit of Rs. 60, determine the number of units of each item that the company should produce to maximize profit. What is the maximum profit?

Answer:



Items	Total items present in store		Profit per unit
	9gms	8 gms	
	Gold (gms)	Copper (gms)	
P	3	1	Rs. 50
Q	1	2	Rs. 60

Let x_1 = number of units of item P
x_2 = number of units of item Q
$Z = 50\ x_1 + 60\ x2$
Such that $3x_1 + x2 \leq 9$
$x_1 + 2x_2 \leq 8$
Such that $x_1, x_2 \geq 0$
From graph, maximum profit
$Z = 2 \times 50 + 3 \times 60$
$= 100 + 180$
No. of units of item P = 2
No. of units of item Q = 3

b. At the beginning of each quarter, a sum of Rs, 1,500 is deposited into a savings account that pays 12% per annum compounded quarterly. Find the amount in the account at the end of four years.

Answer:

$i = 12\%\ p.a = \frac{12}{100} \times \frac{1}{4} = 0.03$

a = Rs. 1,500, n = 16

$A = \frac{a}{i}\{(1+i)^n - 1\}$

$= \frac{1500}{0.03}\{(1+0.03)^{16} - 1\}$

= 50000 [(1.03)16 -1]
= 50000 x 1.604706439
= Rs. 30235.32

Question 14 [5+5=10]

a. A bill of exchange for Rs 722 was drawn on the 3rd April 2009, possible three months after date. It was discounted on 15th April 2009 at 4.75% per annum. What was the discounted value of the bill?

***Answer*:**
Bill value = Rs. 722
Bill is drawn on 3rd april 2009
Bill is due on 3rd July, 2009
Bill is discounted on 15th April, 2009.

$A = \text{Rs. } 722,\ n = \frac{82}{365},\ i = \frac{4.75}{100} = \frac{19}{400}$

Banker's discount = an i $= 722 \times \frac{82}{365} \times \frac{19}{400} = \text{Rs. } 7.70$

$\therefore$ Discounted value of the bill = 722 – 7.70 = Rs. 714.30

b. The cost of manufacturing of certain items consists of Rs 1600 as overheads, Rs 30 per2 for x items produced. How item as the cost of the material and the labour cost Rsx^2/100 many items must be produced to have a minimum-average cost ?

The average cost function AC for a commodity is given by AC = $x + 5 + \frac{36}{x}$ in terms of output x.

Find the:

(i) Total cost and the marginal cost as the function of x.

Answer:

Total cost function C = Ac. x = $x^2 + 5x + 36$

The marginal cost function MC = $\frac{dC}{dx} = \frac{d}{dx}(x^2 + 5x + 36) = 2x + 5$

(ii) Output for which AC increases

Answer:

$\frac{d}{dx} = (x + 5 + \frac{36}{x}) > 0$

$\Rightarrow 1 - \frac{36}{x2} > 0$

$\Rightarrow x^2 - 36 > 0$

$\Rightarrow x > 6$

Hence, the average cost increases, if the output x > 6

Question 15 [3+5=8]

a. The index number for the following data, for the year 2008, taking 2004 as base year was found to be 116. The simple aggregate method was used for calculation. Find the numerical values of x and y if the sum of the prices in the year 2008 is Rs. 2003.

Commodity	Price in Rs. In the year 2004	Price in Rs. In the year 2008
A	20	25
B	10	30
C	30	15
D	25	45
E	X	35
F	50	y

Answer:

$\sum P_0 = 20 + 10 + 30 + 25 + x + 50 = 135 + x$

$\sum P_1 = 25 + 30 + 15 + 45 + 35 + y = 150 + y$

$\Rightarrow 203 = 150 + y \Rightarrow y = Rs.\ 53$

Now simple Aggregate Index = $\frac{\sum P_1}{\sum P_0} \times 100$

$\Rightarrow 116 = \dfrac{203 \times 100}{135 + x}$

$\Rightarrow$ 116x = 4640

$\therefore$ x = 40

Hence x = Rs. 40, y = Rs. 53

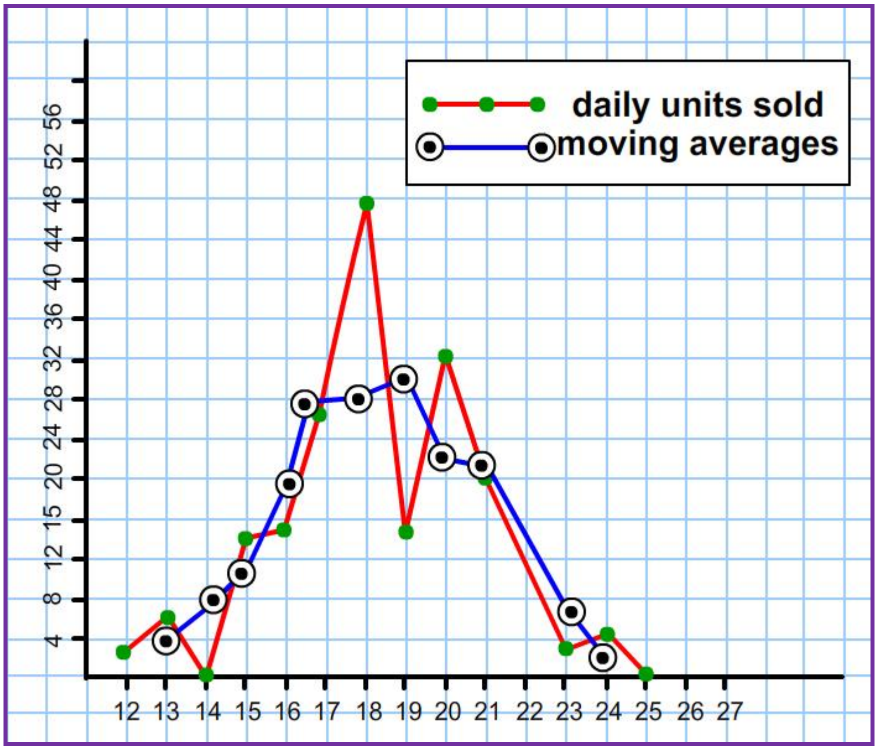
b. Consider the following data:

Dates in the month of april	12	13	14	15	16	17	18	19	20	21	22	23	24	25
Number of units sold	2	5	0	12	13	25	45	13	31	18	11	2	3	1

Calculate three days moving averages and display these and the original figures on the same graph.

Answer:

Date in the month of April	Number of units sold	Three days moving total	Three days moving average
12	2	-	-
13	5	17	2.33
14	0	17	5.66
15	12	25	8.33
16	13	50	16.66
17	25	83	27.66
18	45	83	27.66
19	13	89	29.66
20	31	62	20.66
21	18	60	20
22	11	31	10.33
23	2	16	5.33
24	3	6	2
25	1	-	-


daily units sold
moving averages
4
8
12
15
20
24
28
32
36
40
44
48
52
56
12
13
14
15
16
17
18
19
20
21
22
23
24
25
26
27