2014

Section A *(Alternatives are to be noted)*

Question: 1

[1x10=10]

a.

Two dice are thrown simultaneously. The probability of sum 4 of two numbers so obtained is

i. $\frac{1}{12}$

ii. $\frac{1}{36}$

iii. $\frac{1}{18}$

iv. $\frac{1}{9}$

Answer:

$\frac{1}{9}$

If two dice are thrown, total amount = $6 \times 6 = 36$

The probability of sum of 4 of two dice = 3

$\therefore$ Probability $= \frac{4}{36} = \frac{1}{9}$

b. The condition for the expansion of $(2-3x)^{\frac{1}{4}}$ is

i. $|x| > \frac{2}{3}$

ii. $|x| > \frac{3}{2}$

iii. $|x| < \frac{2}{3}$

iv. $|x| < \frac{3}{2}$

Answer:

$(2-3x)^{\frac{1}{4}}$ if its expansion 2-3x must be greater than zero.

$2 - 3x > 0$

$2 > 3x \quad \because |x| < \frac{2}{3}$

$\frac{2}{3} > x$

OR

Let A = {2, 3, 4, 5, 6, 7, 8, 9}. Let R be the relation on A defined by {(x, y) : x $\in$ A, y $\in$ A and x divides y}. Find domain and range of R.

Answer:

The relation R is $R = \begin{Bmatrix} (2,2),(2,4),(2,6),(2,8),(3,3),(3,6), \\ (3,9),(4,4),(4,8),(5,5),(6,6),(7,7), \\ (8,8),(9,9) \end{Bmatrix}$

Domain of R = { 2,3,4,5,6,7,8,9}
= A

Range of R = { 2,3,4,5,6,7,8,9}
= A

c. Which of the following is the equation of the parabola with vertex at the origin and directrix y = 2?
   i. $y^2 = 8x$
   ii. $y^2 = -8x$
   iii. $x^2 = 8y$
   iv. $x^2 = -8y$

Answer:
Directrix y = 2
So equation $x^2 = -4ay$ here a = 2
$x^2 = -8y$

OR

Fill in the blank:
The eccentricity of the hyperbola $9x^2 - 4y^2 + 36 = 0$ is ________________

Answer:
$9x^2 - 4y^2 + 36 = 0$
$36 = 4y2 - 9x^2$

or, $\frac{4y^2}{36} - \frac{9x^2}{36} = 1$

or, $\left(\frac{y}{3}\right)^2 - \left(\frac{x}{2}\right)^2 = 1$

Eccentricity $= \sqrt{1 + \frac{2^2}{3^2}} = \sqrt{\frac{9+4}{3^2}} = \frac{\sqrt{13}}{3}$

d. If a line makes angle 90°, 60° and 30° with the positive direction of x, y and z respectively, find its direction consines.

Answer:
Let the dc's of the lines be l, m, n. Then
$l = \cos 90^0$
$= 0$
$m = \cos 60^O$
$= \frac{1}{2}$
$n = \cos 30^O$
$= \frac{\sqrt{3}}{2}$

e. Fill in the blank:
The value of

$$\int_{-1}^{1}\left(1 + x + 3x^3 + 5x^5 + ... + 99x^{99}\ )\ dx\right)\ \_\_\_\_\_\_\_\_\_\_\_\_\_\_ .$$

Answer:

$$= \left| x + \frac{x^2}{2} + \frac{3x^4}{4} + \frac{5x^6}{6} + ... + \frac{99x^{100}}{100} \right|_{-1}$$

$$= \left(1 + \frac{1}{2} + \frac{3}{4} + \frac{5}{6} + ...\frac{99}{100}\right) - \left(-1 + \frac{1}{2} + \frac{3}{4} + \frac{5}{6} + ...\frac{99}{100}\right) = 2$$

OR

If $\int f(x)dx = \frac{e^x}{2}(\sin x - \cos x)$ then $f(x)$ will be __________________ .

Answer:

$$f(x) = \frac{d}{dx}(\sin x - \cos x)$$

$$= \frac{e^x}{2}(\sin x - \cos x) + \frac{e^x}{2}(\cos x + \sin x) = e^x \sin x$$

f. Which of the following relations is satisfied by the function $f(x) = \int_1^x \frac{dt}{t}$

i. $f(x + y) = f(x) + f(y)$

ii. $f\left(\frac{x}{y}\right) = f(x) - f(y)$

iii. $f(x) = f(x)\, f(y)$

iv. $f\left(\frac{x}{y}\right) = \frac{f(x)}{f(y)}$

Answer:

$$= \left|\log t\right|_1^x = \log x$$

$$f\left(\frac{x}{y}\right) = f(x) - f(y)$$

g. Order and degree of the differential equation. $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^{\frac{1}{3}} + x^{\frac{1}{4}} = 0$ are respectively

i. 2 and 3
ii. and 12
iii. 2 and 6
iv. 1 and 4.

Answer:
Order = 2 and degree = 1

h. A particle moves according to the law $s = t^3 - 9t^2 + 24r$. the distance covered by the particle before it first comes to rest is

i. 10 unit
ii. 16 unit
iii. 20 unit
iv. 24 unit

Answer:

$s = t^3 - 9t^2 + 24t$
$v = 3t^2 - 18t + 24$ $t = 0$ $v = 24$
$a = 6t - 18$ t:o $a = -18$
$0 = 24 - (6t - 18)\ t$
6t2 – 18t – 24 = 0
Or, $t^2 - 4t + t\text{-}t = 0$
Or, $t^2(t\text{-}t) + (t\text{-}t) = o$
$t = 4$ $t = -1$
$\therefore\ t = 4$
s = 43 – 9 x 42 + 24 × 4 = 16

i. State, whether the following statement is True or False:

$f(x)=|x|$ has no minimum value.

Answer:

$f(x)=|x|$ has a minimum value and that is o. So the following statement is false.

Section B

Question: 2

a. Answer any two questions:

i. Without expanding show that

$$\begin{vmatrix} a & b & c \\ x & y & z \\ yz & zx & xy \end{vmatrix} = \begin{vmatrix} ax & by & cz \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix}$$

Answer:

$$\begin{vmatrix} a & b & c \\ x & y & z \\ yz & zx & xy \end{vmatrix} = \frac{1}{xyz}\begin{vmatrix} ax & by & cz \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix}$$ [According to the three levels 1st level= x, 2nd level = y, 3rd level = z]

$$= \frac{xyz}{xyz}\begin{vmatrix} ax & by & cz \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix}$$ [Third row xyz]

$$= \begin{vmatrix} ax & by & cz \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix}$$

ii. Let f = {(1,3),(2,1),(3,2)} and g = {(1,2), (2,3), (3,1)}, then find (gof) (1).

Answer:
(gof)(x) = g(f(x))
$\therefore$ (gof) 91)
= g(f(1))
= g(3) = 1

iii. Let $f : R \to R$ be defined by $f(x) = 3x - 2$ and $g : R \to R$ be defined by $g(x) = \frac{x+2}{3}$ show that $fog = I_R$.

Answer:

$$(fog)(x) = f(g(x)) = f\left(\frac{x+2}{3}\right) = 3\left(\frac{x+2}{3}\right) - 2$$

x + 2 -2

$= x = I_R(x)$

$\therefore (fog) = I_R$

b. Answer any one question: [2x1=2]

i. If $A = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$, the verfyat A.A = 1

Answer:

Here $A = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$

$\therefore A. = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$

$\therefore A.A = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$

$= \begin{bmatrix} \cos.^2 a + \sin^2 a & \cos a \sin a - \cos a \sin a \\ \sin a \cos - \sin a \cos a & \sin^2 a + \cos a \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$

Hence A.A = 1.

ii. Find the vertex of the parabola y = -2x^2 + 12x -17.

Answer :

$$\frac{y}{2} = -x^2 + 6x - \frac{17}{2}$$

$$\frac{y}{2} = -(x^2 - 2.3x + 9) + 9 - \frac{17}{2}$$

$\frac{y}{2} = -(x-3)^2 + \frac{1}{2}$
$(x-3)^2 = 4(1-y)$
Vertex = (3,1)

c. Answer any one question: [2x1=2]

i. If $y = x + \frac{x^3}{a^2} + \frac{x^5}{5} + ...\infty$, show that $\frac{dy}{dx} = \frac{1}{1-x^2}$

Answer :

$$\frac{dy}{dx} = 1 + x^2 + x^4 + ...\infty$$

$$\frac{dy}{dx} = \frac{1}{1-x^2}$$

ii. If y = a cos (log x) + b sin (log x), show that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$ where a and b are constants.

Answer :

$y' = -a\sin(\log x)\frac{1}{x} + b\cos(\log x)\frac{1}{x}$
$xy' = b\cos(\log x) - a\sin(\log x)$
$xy'' + Y' = -\frac{1}{x}b\sin(\log x) - \frac{1}{x}a\sin(\log x)$
$x^2y'' + y' = -y$
$x^2y'' + xy' + y = 0$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

d. Answer any one question: [2x1=2]

i. If f is integrable function in the interval [-a,a] then show that $\int_{-a}^{a} f(x)dx = \int_0^a [f(x) + f(-x)]dx$ Using the definition of definite integral as the limit of a sum evaluate $\int_0^{-2} ax\,dx$ where a is a constant.

Answer :

g(n) = x

$$\int_0^{-2} ax\,dx = \int_0^{-2a} x\,dx = \left.\frac{x^2}{2}\right|_a^{-2a} = \frac{(-2a)^2}{2} = 2a^2$$

$$\int_{-a}^{d} f(x)\,dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx$$

$$\int_{-a}^{0} f(x)\,dx = \int_0^a f(-x)dx$$

$$\therefore \int_{-d}^{d} f(x)\,dx = \int_0^a f(-x) + f(-x)dx$$

e. Answer any one question: [2x1=2]

i. Find the differential equation of all circles passing through the origin and having their centers on the x-axis.

Answer :

radius = x_1

$(x-x_1)^2 + y^2 = x_1^2$

$x^2 - 2xx_1 + x_1^2 + y^2 = x_1^2$

here x_1 is a constant

$x^2 + y^2 = 2xx_1$

$2x + 2y\frac{dy}{dx} = 2x_1$ ....(i)

$$x^2 + y^2 = x\left(2x + 2y\frac{dy}{dx}\right)$$

$$x^2 + y^2 = 2x^2 + 2xy\frac{dy}{dx}$$

$$2xy\frac{dy}{dx} = y^2 - x^2$$

ii. Find the equation of the curve which passes through the point (4,3) and the gradient of the tangent to the curve at any point on it is equal to the reciprocal of the ordinate of the point.

Answer :

$\frac{dy}{dx} = \frac{1}{y}$ gradient $\frac{dy}{dx}$

$y^2 = 2x + e$ [if (x,y) = (4,3)

c = 1

$y^2 = 2x + 1$

f. Answer any one question: [2x3=6]

i. If $y = a\log|x| + bx^2 + x$ has extreme values x = -1 and dx = 2, find the values of a and b.

Answer :

$y = a \log |x| + bx^2 + x^2 = b - 1$

b = 3

$$y' = \frac{a}{x} + 2ba + 1$$

$$-a - 6 + 1$$

$$a = -5$$

ii. Find the area of the region bounded by the curve y = (x-1) (5-x) and x-axis.

Answer :

y = (x-1) (5-x)

$$\int_1^5 y\,dx = \int_1^5 (x-1)(5-x)\,dx$$

$$= \int_1^5 (5x - 5 - x^{2)}\,dx$$

$$= \int_1^5 (6x - 5 - x^{2)}\,dx$$

$$= \left| \frac{6x^2}{2} - ex - \frac{x^3}{3} \right|_1^5$$

= 8.33 + 2.33 = 10.66

iii. Find the values of x for which the function f(x) = x3-7x^2 + 8x – 10 is monotonic increasing.

Answer :

$f'(x) = 3x^2 - 14x + 8 > 0$

$3x^2 - 14x + 8 > 0$

$3x^2 - 12x - 2x + 8 > 0$

$3x(x - 4) - 2(x - 4) > 0$

$(x-4)(3x-2) > 0 \qquad x > 4 \;\; x > \frac{2}{3} \left(\frac{2}{3}, \infty\right)$

iv. Find the principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$.

Answer:

Let $x = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$\cos x = \frac{\sqrt{3}}{2}$

$\cos x = \cos\frac{\pi}{6}$

$x = \frac{\pi}{6}$

Hence principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is $\frac{\pi}{6}$.

v. A particle moves in a straight line and its velocity v at time t seconds is given by v = ($6t^2$ – 2t + 3) cm/sec. Find the distance travelled by the particle during the first 5 seconds after the start.

Answer :

$v = 6t^2 - 2t + 3$

$A = 12t - 2$

$S = vt + \frac{1}{2}at2$

$= 6t^2 - 2t^2 + 3t + 6t^3 - t2$

$= 12t^3 - 3t^2 + 3t$

As t = 5 second

$s = 12 \times 5^3 - 3 \times 5^2 + 3 \times 5$

=1440

Section C

Question: 3

a. Answer any two questions: [4x2=8]

i. Show that the matrix $A = \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}$ satisfies the equation $A^2 - 6A + 17I = 0$ and hence find A^{-1} where I is the identity matrix and 0 is the null matrix of order 2x2.

Answer:

$$A = \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4-9 & -6-12 \\ 6+12 & -9+16 \end{bmatrix} = \begin{bmatrix} -5 & -18 \\ 18 & 7 \end{bmatrix}$$

$$6A = \begin{bmatrix} 12 & -18 \\ 18 & 24 \end{bmatrix} \qquad 17I = \begin{bmatrix} 17 & 0 \\ 0 & 17 \end{bmatrix}$$

$$A^2 - 6A + 17I$$

$$= \begin{bmatrix} 5 & 18 \\ 18 & 7 \end{bmatrix} - \begin{bmatrix} 12 & -18 \\ 18 & 24 \end{bmatrix} + \begin{bmatrix} 17 & 0 \\ 0 & 17 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

$$17A^{-1} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}$$

$$17A^{-1} = \begin{bmatrix} 4 & -3 \\ -3 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{17}\begin{bmatrix} 4 & -3 \\ -3 & 2 \end{bmatrix}$$

ii. Show that the relation R in R defined as R = {(a,b): a≤b} is transitive.

Answer:

Let (a,b) € R and (b,c) € R

∴ (a a≤b) and b ≤ c ⇒ a ≤ c

∴ (a,c) € R Hence R is transitive.

iii. Form the differential equation of the following family of curves: $xy = Ae^x + Be^{-x} + x^2$

Answer:

Given $xy = Ae^x + Be\text{-}x + x^2$

Diff. w.r.t.x $x\frac{dy}{dx} + 1.y = Ae^x - Be^{-x} + 2x$

Again diff. w.r.t.x, X $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+1.\frac{dy}{dx}=Ae^{x}+Be^{-x}+2$

$\Rightarrow x\frac{d^2y}{dx^2}+2\frac{dy}{dx}=xy-x^2+2$ [Using (1)] which is the required diff. equation

b. Answer any two questions: [4x2=8]

i. Evaluate: $\int\frac{1}{3+2\sin x+\cos x}dx$

Answer:

Let, $1=\int\frac{dx}{3+2\sin x+\cos x}$

Put: $\tan\frac{x}{2}=t$

$\Rightarrow \frac{1}{2}\sec^2\frac{x}{2}dx=dt \quad \Rightarrow dx=\frac{2dt}{1+t^2}$

$\therefore 1=\int\frac{2dt\left(1+t^2\right)}{3+2.\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}$

$=\int\frac{2dt}{3\left(1+t^2\right)+4t+1-t^2}=\int\frac{dt}{t^2+2t+2}=\int\frac{dt}{(t+1)^2+1}=\tan^{-1}(t+1)+c=\tan^{-1}\left(1+\text{thn}\frac{x}{2}\right)+c.$

ii. Show that the binary operation * defined by a * b = -a-b, on z not commulative and associative.

Answer:

i. Let a, b $\in z$

$\therefore a-b\neq b-a$

$\therefore a*b\neq b*a$

Hence binary operation * is not commutative.

ii. Let a, b, c $\in z$

$\therefore (a*b)*c=(a-b)*c=a-b-c$

And a*(b*c)=a*(b-c) = a – (b-c) = a – b + c

$\therefore (a*b)*c\neq a*(b*c)$ Hence binary operation * is not associative.

iii. If $f:R\to R$ be given by $f(x)=\left(3-x^3\right)^{\frac{1}{3}}$, then find the value of fof(x).

Answer:

For all $x \notin R$,

fof(x)= f(f(x))

$$= f\left(3-x^3\right)^{\frac{1}{3}} = \left[3-\left(\left(3-x^3\right)\frac{1}{3}\right)^3\right]^{\frac{1}{3}} = \left[3-\left(3-x^3\right)\right]^{\frac{1}{3}} = \left(x^3\right)^{\frac{1}{3}}$$

$\therefore$ (fof)=x

c. Answer any two questions: [4x2]=8

i. If A ij is the cofactor of the element ij a of the determinant, $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$ then write the value of $a_{32}.A_{32}$.

Answer:

$$\text{Let } A = \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$$

$$A_{32} = \begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix} = -8 \quad (8-30) = 22$$

$a_{32} = 5$

Thus, $a_{32}.A_{32} = 22 \times 5 = 110$

ii. P and Q are two points with position vectors $\overrightarrow{3a} - \overrightarrow{2b}$ and $\vec{a} + \vec{b}$ respectively. Write the position vector of a point R which divides the line segment PQ in the ratio 2:1 externally.

Answer:

Position vector of point R $\overrightarrow{3a} - \overrightarrow{2b}$ and $\vec{a} + \vec{b}$

$$= \frac{2\left(\vec{a}+\vec{b}\right) - 1\left(\overrightarrow{3a} - \overrightarrow{2b}\right)}{2-1}$$

$$= \overrightarrow{2a} + \overrightarrow{2b} - \overrightarrow{3a} + \overrightarrow{2b}$$

$$= \vec{a} + \overrightarrow{4b}$$

iii. For what value of x, is the matrix A = , $\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ x & -3 & 0 \end{bmatrix}$ a skew symmetric matrix?

Answer:

We know that,

$x = a_{31}$

Given matrix is skew symmetric.

Thus, aij = -aji

$\therefore$ $x = a_{31} - a_{13}$

x = -(-2)

x = 2

d. Answer any two questions: [4x2]=8

i. Find the values of a and b for which the following holder: $\begin{bmatrix} a & b \\ -a & 2b \end{bmatrix}\begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

Answer:

Here given, $\begin{bmatrix} a-b \\ -a \ \ 2b \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ $\begin{bmatrix} 2a-b \\ -2a-2b \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

By definition of equality of matrices

2a-b = 5 (i)

-2a –2b = 4 (ii)

On adding (i) and (ii), we get -3b = 9 $\Rightarrow$ b= -3

Putting b= -3 in eq. (i), we get 2a+3 = 5 $\Rightarrow$ a =1

ii. If $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, prove that $A^3 - 4A^2 + A = 0$.

Answer:

$A= \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$

$\Rightarrow A^3 = A^2 A = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 26 & 45 \\ 15 & 26 \end{bmatrix}$

$\therefore A^3 - 4A^2 + A = \begin{bmatrix} 26 & 45 \\ 15 & 26 \end{bmatrix} - 4\begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 26-28+2 & 45-48+3 \\ 15-16+1 & 26-28+2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0.$

iii. A balloon which always remains spherical is being inflated by pumping in gas at the rate of 900 cm ^{3} /sec. Find the rate at which the radius of the balloon is increasing when the radius of the balloon is 15cm

Answer:

Let r be the radius and V be the volume of the balloon. Then $V = \frac{4}{3}\pi r^3$

Diff.w.r.t $\frac{dv}{dt} = 4\pi r^2 \frac{dr}{dt}$ Given: $\frac{dv}{dt}$ =900 cm3/sec where r = 15 cm. we have $900 = 4\pi\,(15)^2 \frac{dr}{dt}$

$\Rightarrow \frac{dr}{dt} = \frac{1}{\pi}$ cm/sec $= \frac{7}{22}$ cm/sec

e. Answer any two questions: [5x2=10]

i. Bicycles and the second plant 40%. 80% of the bicycles are rated of standard quality of the first plant and 90% of standard quality at the second plant. A bicycle is picked up at random and found to be of standard quality. Find the probability that it comes from the second plant.

Answer:

Let A_1 and A_2 be the event that a bicycle is manufactured at plant I and plant II. Let E be the event that the bicycle chosen is of standard quality. Then

$$P(A_1) = = \frac{\frac{4}{10} x \frac{9}{10}}{\frac{6}{10} x \frac{8}{10} + \frac{4}{10} x \frac{9}{10}} = \frac{4 x 9}{6 x 8 + 4 x 9} \quad \frac{60}{100} = \frac{6}{10}$$

$$P(A_2) = \frac{40}{100} = \frac{4}{10}$$

$$P(E / A_1) = \frac{80}{100} = \frac{8}{10}$$

$$P(E / A_2) = \frac{90}{100} = \frac{9}{10}$$

By Bayes' theorem

$$P(A_2 / E) = \frac{P(A_2)P(E / A_2)}{P(A_1)P(E / A_1) + P(A_2)P(E / A_2)}$$

$$= \frac{\frac{4}{10} x \frac{9}{10}}{\frac{6}{10} x \frac{8}{10} + \frac{4}{10} x \frac{9}{10}} = \frac{4 x 9}{6 x 8 + 4 x 9} = \frac{3}{7}$$

ii. Six coins are tossed simultaneously. Find the probability of getting
   a. 3heads
   b. no head
   c. at least one head

Answer.

Let p = probability of getting a head in the a coin

$$\therefore P = \frac{1}{2} \Rightarrow q = 1 - p = \frac{1}{2}$$

Let X = No. of successes in the experiment X can take values 0,1,2,3,4,5,6

Here n = 6. Now, P (X = r) = ${}^{n}C_{r}\, p^{r}\, q^{n-r}$

$$(i) P(X - 3) = {}^{6}C_{3} \left(\frac{1}{2}\right)^{3} \left(\frac{1}{2}\right)^{6-3} = \frac{20}{2^6} = \frac{20}{64} = \frac{5}{16}$$

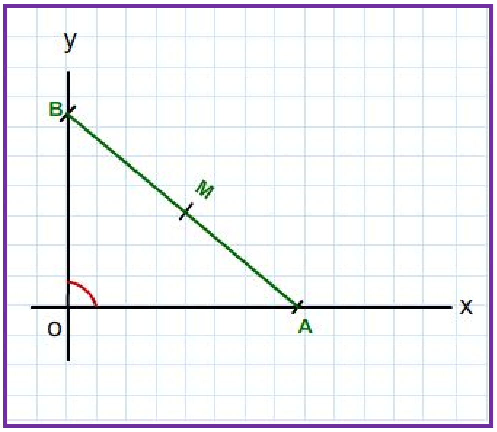
$$(i)\ P(X - 0) = {}^{6}C_{0} \left(\frac{1}{2}\right)^{0} \left(\frac{1}{2}\right)^{6-0} = \frac{1}{64}$$

$$(ii) P(\text{at least one head}) = 1 - P(\text{No head}) = 1 - P\ (X - 0) = 1 - \frac{1}{64} = \frac{63}{64}$$

iii. Using vectors, prove that the mid-point of the hypotenuse of a right-angled triangle is equidistant from its vertices.

Answer:

Let the given right angled triangle be OAB. Take OA and OB as the coordinate axes respectively. Let M be the mid point of the hypotenuse AB. Suppose p.v.s of O,A and B be $\vec{o},\vec{a},\vec{b}$ respectively. Now.



$\overrightarrow{OA} \perp \overrightarrow{OB} \Rightarrow \vec{a},\vec{b}=0$

The position vector of M is $\frac{\vec{a}+\vec{b}}{2}$ Now $\overrightarrow{OM}=\frac{\vec{a}+\vec{b}}{2} \Rightarrow |\overrightarrow{OM}|=\left|\frac{\vec{a}+\vec{b}}{2}\right|$

$\Rightarrow |\overrightarrow{OM}| 2=\frac{1}{4}|\vec{a}+\vec{b}|^2$

$=\frac{1}{4}(\vec{a}-\vec{b}).(\vec{a}+\vec{b})$

$=\frac{1}{4}(a^2+b^2)\left[\because \vec{a}.\vec{b}=0\right]$

Next $\overrightarrow{AM}=\overrightarrow{OM}-\overrightarrow{OA}=\frac{1}{2}(\vec{a}+\vec{b})$

$=\frac{1}{2}(\vec{b}-\vec{a})$

$\Rightarrow |\overrightarrow{AM}|^2=\frac{1}{4}(\vec{b}-\vec{a})^2$

$=\frac{1}{4}(\vec{b}-\vec{a}).(\vec{b}-\vec{a})$

$=\frac{1}{4}(b^2+a^2)=\frac{1}{4}(b^2+a^2)$

Similarly $|\overrightarrow{AM}|^2=\frac{1}{4}(a^2+b^2)$

From(1),(2)and (3), $|\overrightarrow{OM}|^2=|\overrightarrow{AM}|^2=|\overrightarrow{BM}|^2$

$\Rightarrow |\overrightarrow{OM}|=|\overrightarrow{AM}|^2=|\overrightarrow{BM}|$

iv. Observe the inverse of the following matrix using elementary operations. $A=\begin{bmatrix}0 & 1 & 2\\1 & 2 & 3\\3 & 1 & 1\end{bmatrix}$

Answer:

Since A = IA

or $\begin{bmatrix}0 & 1 & 2\\1 & 2 & 3\\3 & 1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}A$

Applying R_1 « R_2 $\begin{bmatrix}1 & 2 & 3\\0 & 1 & 2\\3 & 1 & 1\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}A$

Applying $R_3 \to R_3 - R_1$

$\begin{bmatrix}1 & 2 & 3\\0 & 1 & 2\\0 & -5 & -8\end{bmatrix}=\begin{bmatrix}-2 & 1 & 0\\1 & 0 & 0\\0 & -3 & 1\end{bmatrix}A$

Applying $R_1 \to R_1 - 2R_2$ $\begin{bmatrix}1 & 0 & -1\\0 & 1 & 2\\0 & 0 & 2\end{bmatrix}=\begin{bmatrix}-2 & 1 & 0\\1 & 0 & 0\\0 & -3 & 1\end{bmatrix}A$

Applying $R_1 \to \frac{1}{2}R_3$ $\begin{bmatrix}1 & 0 & -1\\0 & 1 & 2\\0 & 0 & 1\end{bmatrix}=\begin{bmatrix}-2 & 1 & 0\\1 & 0 & 0\\\frac{5}{2} & -\frac{3}{4} & \frac{1}{2}\end{bmatrix}$

v. Write a vector in the direction of $\vec{a} = 2\hat{i} - 6\hat{j} + 3\hat{k}$.

Answer:

Here $\vec{a} = 2\hat{i} - 6\hat{j} + 3\hat{k}$.

$\left|\vec{a}\right| = \sqrt{(2)^2 + (-6)^2 + (3)^2}$

$= \sqrt{4 + 36 + 9}$

$= \sqrt{49}$

$= 7$

$\therefore$ unit vector $\hat{a} = \frac{\vec{a}}{\left|\vec{a}\right|} = \frac{2\hat{i} - 6\hat{j} + 3\hat{k}}{7} = \frac{2\hat{i}}{7} - \frac{6\hat{j}}{7} + \frac{3\hat{k}}{7}$.

vi. If matric A = $(1,2,3,)$, write AA"where A'is the transpose of matrix A.

Answer:

Here $A = (1,2,3,) \setminus A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$\therefore AA = [123]\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = [1+4+9] = [14]$

vii. The money to be spent for the welfare of the employees of a firm is proportional to the rate of change of its total revenue (marginal revenue). If the total revenue (in rupees) received from the sale of x units of a product is given by R(x) = 3x2 + 36x + 5, find the marginal revenue, when x = 5, and write which value does the question indicate.

Answer:

Total revenue, R(x) = 3x2 + 36x + 5

Marginal revenue, $\frac{dR\ (x)}{dx} = 6x + 36$

At x = 5,

$\frac{dR\ (x)}{dx} = 6x + 36 = 66$

Thus, marginal revenue = 66

Section D

Question: 3

Answer the following questions as per instructions:
{Answer any one question from (a) and (b)}
{Answer any one question from (c) and (d)}
{Answer any one question from (e),(f), (g) and (h)}

a.

i. Find the slop of the tangent to the curve y $y = x^3 - x$ at x =2.

Answer:

Here $y = x^3 - x$

$\therefore \frac{dy}{dx} = 3x^2 - 1$

Slope of the tangent at x = 2 is

$m = \left[\frac{dy}{dx}\right]_{at\ x=2} = 3(2)^2 - 1 = 12 - 1 = 11$

ii. Form the differential equation corresponding to $y^2 = a\left(b - x^2\right)$ where a and b are arbitrary constants.

Answer:

$y^2 = a(b-x^2)$

Diff. W.r.t.x $2y\frac{dy}{dx} = a(0-2x) = -2ax \rightarrow y\frac{dy}{dx} = -ax$ (1)

Again diff. W.r.t.x $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -a$ (2)

From(1) and (2), $y\frac{dy}{dx} = x\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right]$ which is the required different equation

b.

i. Show that the function: $f(x) = \left\{\frac{\sin x}{2^x} + \cos x, x \neq 0, x = 0\right.$ is continuous at $x = 0$

Answer:

$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{\sin x}{x} + \cos x\right) = 1+1=2$

Also $f(0) = 2$

$\therefore \lim_{x\to 0} f(x) = f(0)$

Hence $f(x)$ is continuous at $x = 0$

ii. Find the value of K so that the function $f(x) = \left\{Kx^2, \text{ if } x \le 2 \;\; 3, \text{ if } x > 2\right.$ is continuous at x=2

Answer:

$\therefore f(x)$ is continuous at $x = 2$

$\lim_{x\to 2-} f(x) = \lim_{x\to 2+} f(x) = f(2)$

$\therefore \lim_{x\to 2-} Kx^2 = \lim_{x\to 2+} 3$

$K(2)^2 = 3$

$4K = 3$

$\Rightarrow K = \frac{3}{4}$

c.

i. Is the binary operation $*$, defined or set N, given by $a * b = \frac{a+b}{2}$ for all $a, b \in Q$, commutative?

Answer:

$\therefore a * b = \frac{a+b}{2}$

$a * b = \frac{b+a}{2}$ $[\because a, b \in Q]$

$\therefore a * b = b * a$

Hence $*$ is commutaive

ii. Is the above binary operation * associative?

Answer:

Let $a, b, c \in Q$

$$a * (b * c) = a * \left(\frac{b+c}{2}\right)$$

$$= \frac{a + \frac{b+c}{2}}{2}$$

$$= \frac{2a+b+c}{2.2}$$

$$= \frac{2a+b+c}{4}$$

and $$(a * b) * c = \left(\frac{a+b}{2}\right) * c$$

$$= \frac{\frac{a+b}{2} + c}{2}$$

$$= \frac{a+b+2c}{2.2}$$

$$= \frac{a+b+2c}{4}$$

$$\therefore a * (b * c) \neq (a * b)c$$

Hence * is not associative.

d.

i. For what value of x, the matrix $\begin{bmatrix} \frac{5-x}{2} & \frac{x+1}{4} \end{bmatrix}$ is singular?

Answer:

Let A = $\begin{bmatrix} \frac{5-x}{2} & \frac{x+1}{4} \end{bmatrix}$

It is given that the matrix A is singular, therefore $|A| = 0$

$\Rightarrow \begin{bmatrix} \frac{5-x}{2} & \frac{x+1}{4} \end{bmatrix}$

$\Rightarrow 4(5 - x) - 2(x + 1) = 0$

$\Rightarrow 20 - 4x - 2x - 2 = 0$

$\Rightarrow -6x + 18 = 0$

$\Rightarrow x = \frac{-18}{-6} = 3$

Thus, when x = 3, the given matrix A is singular.

ii. Write A-1 for A = $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

Answer:

$$A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{1}{[(2x3)-(1x5)]}\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{1}{(6-5)}\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

e.

i. Evaluate: $\int x\sqrt{x^4 - 1}dx$

Answer:

Let I = $\int x\sqrt{x^4 - 1}dx$

Put $x^2 = t \Rightarrow 2x\,dx = dt$

$$\therefore I = \frac{1}{2}\int \sqrt{t^2 - 1}\,dt$$

$$= \frac{1}{2}\left[\frac{t\sqrt{t^2-1}}{2} - \frac{1}{2}\log\ (t + \sqrt{t^2-1}\ )\right] + c$$

$$= \frac{1}{4}\left[\sqrt{t^2-1}\log\quad (t + \sqrt{t^2-1}\right] + c$$

$$= \frac{1}{4}\left[\ x^2\sqrt{x^4-1}\log\quad (x^2 + \sqrt{x^4-1}\ \right] + c$$

ii. Discuss the continuity of the function: $F(x)\begin{Bmatrix} x, \text{ if } 0 \le x < \frac{1}{2} \\ \frac{1}{2}, \text{ if } x = \frac{1}{2} \\ 1-x, \text{ if } x ? \frac{1}{2} \end{Bmatrix}$ at x = $\frac{1}{2}$

Answer:

$$\text{LHL} = \lim_{x\to\frac{1}{2}^-} f(x) = \lim_{x\to\frac{1}{2}^-} x = \lim_{h\to 0}\left(\frac{1}{2}-h\right) = \frac{1}{2}$$

$$\text{RHL} = \lim_{x\to\frac{1}{2}^-} f(x) = \lim_{x\to\frac{1}{2}^-} (1-x) = \lim_{h\to 0}\left(1-\left(\frac{1}{2}-h\right)\right) = \lim_{h\to 0}\left(\frac{1}{2}-h\right) = \frac{1}{2}$$

Also $f\left(\frac{1}{2}\right)=\frac{1}{2}$

$\therefore$ LHL = RHL = $f\left(\frac{1}{2}\right)$

iii. If $y=\cos^{-1}\left(\frac{3x+4\sqrt{1-x^2}}{5}\right)$, find $\frac{dy}{dx}$

Answer:

$y=\cos^{-1}\left[\frac{3}{5}+\frac{4}{5}\sqrt{1-x^2}\right]$

$=\cos^{-1}\left[\frac{3}{5}.\cos\theta+\frac{4}{5}\sin\theta\right]$ where $x=\cos\theta$

$=\cos^{-1}[\cos\alpha.\operatorname{Cos}\theta+\sin\alpha.\operatorname{Sin}\theta]$ Q if $\frac{3}{5}=\cos\alpha$, then $\frac{4}{5}=\sin\alpha$

$=$ cos-1 $[\cos(\alpha-\theta)]=\alpha-\theta=\cos^{-1}(3/5)-\cos^{-1}x$

$\Rightarrow\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}$

f.

i. A bag contains 4 balls two balls are drawn at random, and are found to be white. What is the probability that all balls are white?

Answer:

E_1: Bag contains 2 white balls and 2 non whites

E_2: Bag contains 3 white balls and 1 non whites

E_3: Bag contains 4 white balls

A: Getting two white balls

$P(E_1)=\frac{1}{3}$, $P(E_1)=\frac{1}{3}$, $P(E_1)=\frac{1}{3}$,

$P(E_1)=\frac{2c_2}{4c_2}=\frac{1}{6}$, $P(\frac{A}{E_2})=\frac{3c_2}{4c_2}=\frac{1}{2}$, $P(\frac{A}{E_2})=1$

$$P(E_3/A\frac{E_3}{A})=\frac{P(E_3).P(\frac{A}{E_3})}{P(E_1)P(\frac{A}{E_1})+P(E_2)P(\frac{A}{E_2})+P(E_3)P(\frac{A}{E_3})}$$

$$=\frac{\frac{1}{3}.1}{\frac{1}{3}.\frac{1}{6}+\frac{1}{3}.\frac{1}{2}+\frac{1}{3}.1}$$

$$=\frac{6}{10}$$

$$=\frac{3}{5}$$

ii. One kind of cake requires 300 g of flour and 15 g of fat, another kind of cake requires 150 g of flour and 30 g of fat. Find the maximum number of cakes which can be made from 7.5 kg of flour and 600g of fat, assuming that there is no shortage of the other ingredients' used in making the cakes. Make it as an L.P.P and solve it graphically.

Answer:

Let x cakes of first type and y cakes of second type are made

Maximize $S = x + y$

subject to $300x + 150y \le 7500$ or $2x + y \le 50$

$15x + 30y \le 600$ or $x + 2y \le 40$

$x \ge 0, y \ge 0$

Vertices of feasible region are A(0,20), B(20,10) C(25,0)

Maximum cakes = 20 + 10 = 30

g.

i. Solve the following differential equation:

$$(x^2 - 1)\frac{dy}{dx} + 2xy = \frac{1}{x^2 - 1}; |x| \ne 1$$

Answer:

Given differential equation can be written as

$$\frac{dy}{dx} + \frac{2xy}{x^2 - 1}.y = \frac{1}{x^2 - 1}$$

$$= \frac{1}{(x^2 - 1)^2}$$

Which is of the form $\frac{dy}{dx} + P(x).y = Q(x)$

$$\int P(x)\,dx = \int \frac{2x}{x^2 - 1}dx = \log\left|x^2 - 1\right|$$

$\therefore$ Integrating factor $= e^{\int P(x)\,dx} = (x^2 - 1)$

$\therefore$ The solution is $(x^2 - 1).y = \int \frac{1}{(x^2 - 1)^2}(x^2 - 1)dx$

$$(x^2 - 1).y = \frac{1}{2}\log\left|\frac{x - 1}{x + 1}\right| + c$$

ii. Find the value of k if the function $f(x) = \begin{cases} kx^2, x \ge 1 \\ 4, x < 1 \end{cases}$ is continuous at x = 1.

Answer:

$\because$ f(x) is continuous at x = 1

$$\therefore \lim_{x \to 1} f(x) = f(1)$$

$$\therefore \lim_{x \to 1+} kx^2 = 4$$

Put $x = 1 + h, h \to 0$

$$\lim_{h \to 0} k\ (1+h)^2 = 4$$

$k\ (l)^2 = 4 \Rightarrow\ k = 4$

h.

i. Find the projection of $\vec{b}+\vec{c}$ on $\vec{a}$ where $\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$.

Answer:

$\vec{b}+\vec{c} = \hat{i}+2\hat{j}-2\hat{k}+2\hat{i}-\hat{j}+4\hat{k} = 3\hat{i}+\hat{j}+2\hat{k}$

Projection of $\vec{b}+\vec{c}$ on $\vec{a} = \dfrac{(\vec{b}+\vec{c}\ ).\vec{a}}{|\vec{a}|}$

$= \dfrac{(3\hat{i}+\hat{j}+2\hat{k}\ )\ (2\hat{i}-2\hat{j}+\hat{k})}{\sqrt{4+4+1}}$

$= \dfrac{6-2+2}{3}$

$= \dfrac{6}{3}$

$= 2$

ii. Show that the curves $xy = a^2$ and $x^2 + y^2 = 2a^2$ touch each other

Answer:

The given curves are $xy = a^2$

$x^2 + y^2 = 2a^2$ (1) and (2) intersect (a,a) and (a,a).

Diff. (1) and (2) to get $x\dfrac{dy}{dx} + 1.y = 0 \Rightarrow \dfrac{dy}{dx} = \dfrac{y}{x}$ and $2x + 2y\dfrac{dy}{dx} = \dfrac{x}{y}$

$\therefore$ At (a,a), $m_1 = \left(\dfrac{dy}{dx}\right)_{(a,a)} = \dfrac{a}{a} = -1$

$m_2 = \left(\dfrac{dy}{dx}\right)_{(a,a)}$

$= \dfrac{a}{a} = -1$

$\Rightarrow \tan\theta = \left|\dfrac{m_1 - m_2}{1+m_1m_2}\right| = \left|\dfrac{-1-(-1)}{1+(-1)(-1)}\right| = 0$

$\Rightarrow \theta = 0$

$\Rightarrow (1)$ and (2) touch at (a,a).

Similarly these at $(-a,-a)$ also.