2016

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Set I

Section A *(Question numbers 1 to 10 carry 1 mark each)*

Question: 1

Write the principal value of $\tan^{-1}\left(\sqrt{3}\right)-\cot^{-1}\left(-\sqrt{3}\right)$

Answer:

Let x = tan-1 $\left(\sqrt{3}\right)$

tan x = $\left(\sqrt{3}\right)$

tan x = tan $\frac{\pi}{3}$

$x=\frac{\pi}{3}$

let $y=\cot^{-1}\left(-\sqrt{3}\right)$

$\cot y=\cot\frac{\pi}{6}$

$\cot y=\cot\left(\pi-\frac{\pi}{6}\right)$

$\cot y=\cot\frac{5\pi}{6}$

$y=\frac{5\pi}{6}$

Now,

$\tan^{-1}\left(\sqrt{3}\right)-\cot^{-1}\left(-\sqrt{3}\right)$

$=\frac{\pi}{3}-\frac{5\pi}{6}$

$=-\frac{\pi}{2}$

Question: 2

Write the value of $\tan^{-1}\left[2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right]$

Answer:

$=\tan^{-1}\left[2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right]$

$=\tan^{-1}\left[2\sin\left\{2\cos^{-1}\left(\cos\frac{\pi}{6}\right)\right\}\right]$

$=\tan^{-1}\left[2\sin\left\{\left(2\frac{\pi}{6}\right)\right\}\right]$

$=\tan^{-1}\left[2\sin\left\{\left(\frac{\pi}{3}\right)\right\}\right]$

$$= \tan^{-1}\left[2\left(\frac{\sqrt{3}}{2}\right)\right]$$

$$= \tan^{-1}\left[\sqrt{3}\right]$$

$$= \tan^{-1}\left[\tan^{-1}\left(\frac{\pi}{3}\right)\right]$$

$$= \frac{\pi}{3}$$

Question: 3

For what value of x, is the matrix A = , $\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ x & -3 & 0 \end{bmatrix}$ a skew symmetric matrix?

Answer:

We know that,

$x = a_{31}$

Given matrix is skew symmetric.

Thus, aij = -aji

$\therefore\ x = a_{31} - a_{13}$

$x = -(-2)$

$x = 2$

Question: 4

If matrix A = $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and A2 = kA , then write the value of k.

Answer:

$$A = A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$$A^2 = 2\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$A^2 = 2A$

Also,

$A^2 = kA$

Thus,

$k = 2$

Question: 5

Write the differential equation representing the family of curves y = mx, where m is an arbitrary constant.

Answer:

We have,

$y = mx$

on differentiation

$$\frac{dy}{dx}=m$$

$$m=\frac{y}{x}$$

The differential equation representing the family of curves y = mx, is
xdy – ydx = 0.

Question: 6

If A ij is the cofactor of the element ij a of the determinant, $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$ then write the value of

$a_{32}.A_{32}$.

Answer:

Let $A=\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$

$A_{32}=\begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix}=-8\ \ (8-30)=22$

$a_{32} = 5$

Thus, $a_{32}.A_{32} = 22 \times 5 = 110$

Question: 7

P and Q are two points with position vectors $\overrightarrow{3a}-\overrightarrow{2b}\text{and}\vec{a}+\vec{b}$ respectively. Write the position vector of a point R which divides the line segment PQ in the ratio 2:1 externally.

Answer:

Position vector of point R $\overrightarrow{3a}-\overrightarrow{2b}\text{and}\vec{a}+\vec{b}$

$$=\frac{2\left(\vec{a}+\vec{b}\right)-1\left(\overrightarrow{3a}-\overrightarrow{2b}\right)}{2-1}$$

$$=\overrightarrow{2a}+\overrightarrow{2b}-\overrightarrow{3a}+\overrightarrow{2b}$$

$$=\vec{a}+\overrightarrow{4b}$$

Question: 8

Find $\left|\vec{x}\right|$, if for a unit vector $\vec{a}\left(\vec{x}-\vec{a}\right).\left(\vec{x}+\vec{a}\right)=15$

Answer:

$$\left(\vec{x}-\vec{a}\right).\left(\vec{x}+\vec{a}\right)=15$$

$$\left|\vec{x}\right|^2-\left|\vec{a}\right|^2-15$$

$$\left|\vec{x}\right|^2=\sqrt{15+\left|\vec{a}\right|^2}$$

$$\left|\vec{x}\right|^2=\sqrt{15+\left|1\right|^2}\quad\left(\because\left|\vec{a}\right|=1\right)$$

$$\left|\vec{x}\right|=4$$

Question: 9

Find the length of the perpendicular drawn from the origin to the plane 2x-3y + 6z + 21 =0.

Answer:

Length of perpendicular = $\left|\dfrac{2(0)-3(0)+6(0)+21}{\sqrt{22+(-3)^2+6^2}}\right|$

$=\dfrac{21}{7}$

=3

Question: 10

The money to be spent for the welfare of the employees of a firm is proportional to the rate of change of its total revenue (marginal revenue). If the total revenue (in rupees) received from the sale of x units of a product is given by R(x) = 3x2 + 36x + 5, find the marginal revenue, when x = 5, and write which value does the question indicate.

Answer:

Total revenue, R(x) = 3x2 + 36x + 5

Marginal revenue, $\dfrac{dR(x)}{dx}=6x+36$

At x = 5,

$\dfrac{dR(x)}{dx}=6x+36=66$

Thus, marginal revenue = 66

Section B *(Question numbers 11 to 22 carry 4 marks each)*

Question: 11

Consider f : $R_+ \rightarrow [4, \infty)$ given by f(x) = x^2+4. Show that f is invertible with the inverse f-1 of f given by $f^{-1}(y)=\sqrt{y-4}$, where R+ is the set of all non-negative real numbers.

Answer:

$f(x) = x^2 + 4$

Let $f(x_1) = f(x_2)$

$x_1^2+4=x_2^2+4$

$x_1 = x_2$

thus, f(x) is one-one since, x2 + 4 is a real number. Thus, for every y in the co-domain, f, there exist a number x in R+ $R_+ \rightarrow [4, \infty)$ such that

f(x) = y = x2 + 4

Thus, we can say that f(x) is onto.

Now, f(x) is one-one and onto. Hence, f(x) is invertible.

Let $f(x) = y = x^2 + 4$

$x = f^{-1}(y)$

$f^{-1}(y)=\sqrt{y-4}$

Question: 12

Show that $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right)=\frac{4-\sqrt{7}}{3}$

Answer:

LHS

Let $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right)=x$

$\frac{1}{2}\sin^{-1}\frac{3}{4}= 2\tan^{-1}x$

$\sin^{-1}\frac{3}{4}= 2\tan^{-1}x$

$\sin^{-1}\left(\frac{2x}{1+x^2}\right)= 2\tan^{-1}x$

We know that,

$\sin^{-1}\frac{2x}{1+x^2}=\frac{3}{4}$

Thus,

$\sin^{-1}\frac{2x}{1+x^2}=\frac{3}{4}$

8x = 3 + 3x2

3x2 – 8x + 3 = 0

$x=\frac{8\pm\sqrt{64-36}}{6}$

$x=\frac{4\pm\sqrt{7}}{3}$

$x=\frac{4-\sqrt{7}}{3}$ = RHS

Thus, LHS = RHS

(OR)

Solve the following equation:

$\cos\left(\tan^{-1}x\right)=\sin\left(\cot^{-1}\frac{3}{4}\right)$

Answer:

$\cos\left(\tan^{-1}x\right)=\sin\left(\cot^{-1}\frac{3}{4}\right)$

$\cos\left(\tan^{-1}x\right)=\sin\left(\tan^{-1}\frac{3}{4}\right)$

$\cos\left(\tan^{-1}\frac{1}{x}\right)= \cos\left[\frac{\pi}{2}\right]-\tan^{-1}\frac{4}{3}$

On comparing

$\tan^{-1}\frac{x}{1}=\frac{\pi}{2}-\tan^{-1}\frac{4}{3}$

$\tan^{-1}\frac{x}{1}+\tan^{-1}\frac{4}{3}=\frac{\pi}{2}$

$\tan^{-1}\left(\frac{\frac{3x+4}{3}}{\frac{3x-4}{3}}\right)=\frac{\pi}{2}$

$\left(\frac{\frac{3x+4}{3}}{\frac{3x-4}{3}}\right)=\frac{\pi}{2}$

$\frac{3x+4}{3x-4}=\infty$

$3-4x = 0$

$$x=\frac{3}{4}$$

Question: 13

Using properties of determinants prove the following:

$$\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix} = 9y^2\ (x+y)$$

Answer:

$$\text{LHS} == \begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$$

$R_1 \to R_1 + R_2 + R_3$

$$= \begin{vmatrix} 3x+3y & 3x+3y & 3x+3y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$$

Taking common from R_1

$$= (3x+3y) \begin{vmatrix} 1 & 1 & 1 \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$3\ (x+y) \begin{vmatrix} 1 & 0 & 0 \\ x+2y & -2y & -y \\ x+y & y & -y \end{vmatrix}$$

Taking common from C_2 and C_3

$$= -3y^2\ (x+y) \begin{vmatrix} 1 & 0 & 0 \\ x+2y & -2y & 1 \\ x+y & y & -y \end{vmatrix}$$

$=-3y^2(x+y[1-(-2-1)]$

$=9y^2(x+y)$

$=$ RHS

Question: 14

If $y^x = e^{y-x}$, prove that $\frac{dy}{dx}=\frac{(1+\log y)^2}{\log y}$

Answer:

$y^x = e^{y-x}$

Taking log both sides

$x \log y = y-x$

$y = x\log y + x$………………….equation (1)

On differentiation

$$\frac{dy}{dx}=x.\frac{dy}{dx}+\log y+1$$

$$\frac{dy}{dx}=\frac{x}{y}.\frac{dy}{dx}+\log y+1$$

$$\left(1-\frac{x}{y}\right)\frac{dy}{dx}=1+\log y$$
$$\left(\frac{y-x}{y}\right)\frac{dy}{dx}=1+\log y$$
$$\frac{dy}{dx}=\frac{y\ \ (1+\log y)}{y-x}$$
Put value of y from equation 1
$$\frac{dy}{dx}=\frac{(x\log y+x)\ (1+\log y)}{x\log y+x-x}$$
$$\frac{dy}{dx}=\frac{(x\log y+1)\ (1+\log y)}{x\log y}$$
$$\frac{dy}{dx}=\frac{(1+\log y)^2}{\log y}$$

Question: 15

Differentiate the following with respect to x:
$$y=\sin^{-1}\left(\frac{2^{x+1}.3^x}{1+(36)^x}\right)$$

Answer:

Let $y=\sin^{-1}\left(\frac{2^{x+1}.3^x}{1+(36)^x}\right)$
$$=\sin^{-1}\left(\frac{2^{x+1}.3^x}{1+(4x9)^x}\right)$$
$$=\sin^{-1}\left[\frac{2\ \ (6^x)}{1+\{(6)^x\}^2}\right]\qquad\left[\because \sin^{-1}\left(\frac{2x}{1+x^2}\right)=2\tan^{-1}x\right]$$
$=2\tan^{-1}(6x)$
Now,
$y=2\tan^{-1}(6x)$
On differentiation

$$\frac{dy}{dx}=\frac{2}{1+(6^x)^2}x\left[(6^x)\log 6\right]$$
$$\frac{dy}{dx}=\frac{(6^x)\log 6}{1+(36)^x}$$
$$\frac{dy}{dx}=\left(\frac{2^{x+1}.3^x}{1+(36)^x}\right)\log 6$$

Question: 16

Find the value of k, for which $f(x)=\begin{cases}\dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & \text{if } -1\le x<0\\[2mm] \dfrac{2x+1}{x-1}, & \text{if } 0\le x<1\end{cases}$ is continuous at x = 0.

Answer:

$$f(x)=\begin{cases}\dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & \text{if } -1\le x<0\\ \dfrac{2x+1}{x-1}, & \text{if } 0\le x<1\end{cases}$$

If f(x) is continuous then,

$$\lim_{x\to 0^{+}}\frac{2x+1}{x-1}=\lim_{x\to 0^{-}}\frac{\sqrt{1+kx}-\sqrt{1-kx}}{x}=\lim_{x\to 0}\frac{2x+1}{x-1}$$

$$\frac{0+1}{0-1}=\lim_{x\to 0^{-}}\frac{\sqrt{1+kx}-\sqrt{1-kx}}{x}\times\left(\frac{\sqrt{1+kx}+\sqrt{1-kx}}{\sqrt{1+kx}+\sqrt{1-kx}}\right)=\frac{0+1}{0-1}$$

$$\lim_{x\to 0^{-}}\frac{1+kx-(1-kx)}{x\left(\sqrt{1+kx}+\sqrt{1-kx}\right)}=-1$$

$$\lim_{x\to 0^{-}}\frac{2kx}{x\left(\sqrt{1+kx}+\sqrt{1-kx}\right)}=-1$$

$$2k\lim_{x\to 0^{-}}\frac{1}{\left(\sqrt{1+kx}+\sqrt{1-kx}\right)}=-1$$

$$2k\frac{1}{\left(\sqrt{1+0}+\sqrt{1-0}\right)}=-1$$

$$\frac{2k}{2}=1$$

$k=-1$

OR

If $x = a\cos^3\theta$ and $y = a\sin^3\theta$, then find the value of $\frac{d^2y}{dx^2}$ at $\theta=\frac{\pi}{6}$

Answer:

$x = a\cos^3\theta$

$$\frac{dx}{d\theta}=3a\cos^2\theta\ (-\sin\theta)$$

$$\frac{dx}{d\theta}=3a\cos^2\theta-\sin\theta$$

$y = a\sin^3\theta$

$$\frac{dx}{d\theta}=3a\sin2\theta\cos\theta$$

$$\frac{\cos2x-\cos2\alpha}{\cos x-\cos\alpha}$$

$$=\frac{-2\sin\left(\frac{2x+2\alpha}{2}\right)\sin\left(\frac{2x-2\alpha}{2}\right)}{-2\sin\left(\frac{x+\alpha}{2}\right)\sin\left(\frac{x-\alpha}{2}\right)}$$

$$=\frac{\sin x+\alpha}{\sin\left(\frac{x+\alpha}{2}\right)\sin\left(\frac{x-\alpha}{2}\right)}$$

$$= \frac{\left[\left\{2\sin\left(\frac{x+\alpha}{2}\right)\cos\left(\frac{x-\alpha}{2}\right)\right\}\left\{2\sin\left(\frac{x-\alpha}{2}\right)\cos\left(\frac{x-\alpha}{2}\right)\right\}\right]}{\sin\left(\frac{x+\alpha}{2}\right)\sin\left(\frac{x-\alpha}{2}\right)}$$

$$= \left\{2\cos\left(\frac{x+\alpha}{2}\right)\right\}\left\{2\cos\left(\frac{x-\alpha}{2}\right)\right\}$$

$$= 4\cos\left(\frac{x+\alpha}{2}\right)\cos\left(\frac{x-\alpha}{2}\right)$$

$$= 2\left[\cos\left(\frac{x+\alpha}{2}\right)+\left(\frac{x-\alpha}{2}\right)+\cos\left(\frac{x+\alpha}{2}\right)+\left(\frac{x-\alpha}{2}\right)\right]$$

$= 2\cos x + 2\cos\alpha$

Now,

$$\int\frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha}.dx$$

$$\int(2\cos x + 2\cos\alpha)dx$$

$= 2\sin x + 2x\cos\alpha + C$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta}$$

$$\frac{dy}{dx} = \tan\theta$$

$$\frac{d^2y}{dx^2} = -\sec^2\theta.\frac{d\theta}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{\sec^2\theta}{3a\cos^2\sin\theta}$$

$$\frac{d^2y}{dx^2} = \frac{1}{3a\cos^4\sin\theta}$$

$$\frac{d^2y}{dx^2} = \frac{1}{3a\sin\frac{\pi}{6}\cos^4\frac{\pi}{6}}$$

$$\frac{d^2y}{dx^2} = \frac{1}{3a\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)^4}$$

$$\frac{d^2y}{dx^2} = \frac{32}{27a}$$

Question: 17 **

Evalutate: $\int\frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha}$

OR

Evaluate: $\int\frac{x+2}{\sqrt{x^2+2x+3}}$

Answer:

$$\int \frac{x+2}{\sqrt{x^2+2x+3}}$$

Let x+ 2 = A $\frac{d}{dx}$ (x2+2x+3) + B

X + 2 = A(2x+2) + B

X+2 = 2Ax + 2A + B

On equating coefficients

X = 2Ax

A = $\frac{1}{2}$

2 = 2A + B

$$2 = 2\left(\frac{1}{2}\right) + B$$

B = 1

Now,

$$\int \frac{x+2}{\sqrt{x^2+2x+3}} dx$$

$$= \frac{1}{2}\int \frac{2\ (x+2)}{\sqrt{x^2+2x+3}} dx$$

$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}} dx + \int \frac{1}{\sqrt{x^2+2x+3}} dx$$

= A+B……………..Equation 1

$$A = \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}} dx$$

Let x^2+2x+3 = t

(2x+2)dx = dt

Now,

$$\frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}} dx$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{t}} dt$$

$$= \frac{1}{2}\ (2\sqrt{t}\,)$$

$$= \sqrt{x^2+2x+3}$$

$$B = \int \frac{2x+2}{\sqrt{x^2+2x+3}} dx$$

$$= \int \frac{1}{\sqrt{x^2+2x+1+2}} dx$$

$$= \int \frac{1}{\sqrt{(x+1)^2 + \left(\sqrt{2}\right)^2}}$$

$$= \log\left| (x+1) + x^2+2x+3 \right|$$

Put value of A & B in equation 1

=A+ B

$$= \sqrt{x^2+2x+3} + \log\left| (x+1) + x^2+2x+3 \right| + C$$

Question: 18

Evaluate: $\int \frac{dx}{x(x^5+3)}$

Answer:

$$\int \frac{dx}{x\ \ (x^5+3\,)}$$

$$= \int \frac{x^4 dx}{x\ \ (x^5+3\,)}$$

Let $(x^5+3) = t$

$5x^4 dx = dt$

$$\frac{1}{5}\int \frac{dt}{t\ \ (t-3\,)}$$

$$\text{Let} \frac{1}{t\ \ (t-3\,)} = \frac{A}{t} + \frac{B}{t\ \ (t-3\,)}$$

$1 = A(t\text{-}3) + Bt$

$1 = At - 3A + Bt$

$1 = (A+B)t - 3A$

On comparing coefficients

$1 = -3A$

$$A = \frac{-1}{3}$$

$A+B=0$

$$A = \frac{-1}{3}$$

$$B = \frac{1}{3}$$

Now,

$$= \frac{1}{5}\int \frac{dt}{t\ \ (t-3\,)}$$

$$= \frac{1}{5}\int \left[\frac{-1}{3t} + \frac{1}{3\ \ (t-3\,)}\right] dt$$

$$= \frac{-1}{15}\log t + \frac{-1}{15}\log\ \ (t-3\,) + C$$

$$= \frac{-1}{15}\left[\log\left(\frac{t}{t-3}\right)\right] + C$$

$$= \frac{-1}{15}\left[\log\left(\frac{x^5+3}{\left(x^5+3\right)-3}\right)\right] + C$$

$$= \frac{-1}{15}\left[\log\left(\frac{x^5+3}{x^5}\right)\right] + C$$

$$= \frac{-1}{15}\log\left[\left(\frac{x^5+3}{x^5}\right)\right] + C$$

Question: 19

Evaluate: $\int_0^{2\pi} \frac{1}{1+e^{\sin x}} dx$

Answer:

Let I = $\int_0^{2\pi} \frac{1}{1+e^{\sin x}} dx$

We know that,

$\int_0^{2a} f(x)\, dx = \int_0^{a} \{f(x) + f(2a - x)\} dx$

$\therefore$ I = $\int_0^{\pi} \theta \left[\frac{1}{1+e^{\sin x}} + \frac{1}{1+e^{\sin(2\pi - x)}}\right] dx$

I = $\int_0^{\pi} \left[\frac{1}{1+e^{\sin x}} + \frac{1}{1+e^{\sin x}}\right] dx$

I= $\int_0^{\pi} \left[\frac{1+e^{\sin x}}{1+e^{\sin x}}\right] dx$

I= $\int_0^{\pi} dx$

I= $\left[x\right]_0^{\pi}$

Question: 20

If $\vec{a} = \hat{i} - \hat{j} + 7\hat{k}$ and $\vec{b} = 5\hat{i} - \hat{j} + \lambda\hat{k}$, then find the value of λ, so that $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ are perpendicular vectors.

Answer:

$\vec{a} = \hat{i} - \hat{j} + 7\hat{k}$

$\vec{b} = 5\hat{i} - \hat{j} + \lambda\hat{k}$

$\vec{a} + \vec{b} = \hat{i} - \hat{j} + 7\hat{k} + 5\hat{i} - \hat{j} + \lambda\hat{k}$

$= 6\hat{i} - 2\hat{j} + (7 + \lambda)\hat{k}$

$\vec{a} - \vec{b} = \hat{i} - \hat{j} + 7\hat{k} - (5\hat{i} - \hat{j} + \lambda\hat{k})$

$= -4\hat{i} + (7 - \lambda)\hat{k}$

Since, $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ are perpendicular vectors

$(\vec{a}+\vec{b})$and$(\vec{a}-\vec{b}) = 0$

$\left[6\hat{i} - 2\hat{j} + (7 + \lambda)\hat{k}\right] . \left[-4\hat{i} + (7 - \lambda)\hat{k}\right] = 0$

$-24 + (7+\lambda)(7-\lambda) = 0$

$-24 + 49 - \lambda^2 = 0$

$\lambda^2 = 25$

$\lambda = \pm 5$

Question: 21

Show that the lines $\vec{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(\hat{i} - 2\hat{j} + 2\hat{k})$ $\vec{r} = 5\hat{i} - 2\hat{j} + \mu(3\hat{i} + 2\hat{j} + 6\hat{k})$ are intersecting. Hence find their point of intersection.

Answer:

Lines are: $\vec{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(\hat{i} - 2\hat{j} + 2\hat{k})$ $\vec{r} = 5\hat{i} - 2\hat{j} + \mu(3\hat{i} + 2\hat{j} + 6\hat{k})$

Position vectors of the given lines are:

$3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(\hat{i} - 2\hat{j} + 2\hat{k}) = (3 + \lambda)\hat{i} + (2 + 2\lambda)\hat{j} + (-4 + 2\lambda)\hat{k}$

$5\hat{i} - 2\hat{j} + \mu(3\hat{i} + 2\hat{j} + 6\hat{k}) = (5 + \mu)\hat{i} + (-2 + 2\mu)\hat{j} + 6\mu\hat{k}$

If these lines are intersecting, then there will be a common point

$(3+\lambda)\hat{i} + (2+2\lambda)\hat{j} + (-4+2\lambda)\hat{k} = (5+\mu)\hat{i} + (-2+2\mu)\hat{j} + 6\mu\hat{k}$
On comparing the coefficients
$3+\lambda = 5 + 3\mu$
$2 + 2\lambda = -2 + 2\mu$
$-4+2\lambda = 6\mu$
On solving above equations
$\lambda = -4,\ \mu = -2$
Position vector of the point of intersection

$$\vec{r} = 3\hat{i} + 2\hat{j} + 6\hat{k} - 4\left(\hat{i} - 2\hat{j} + 2\hat{k}\right) = -\hat{i} - 6\hat{j} - 12\hat{k}$$

Hence, the coordinates of the point of intersection are (-1,-6,-12)
On solving these equations,

$$\frac{a}{8+10} = \frac{b}{5+12} = \frac{c}{6-2}$$

$$\frac{a}{18} = \frac{b}{17} = \frac{c}{4} = \lambda$$

$a = 18\lambda,\ b = 17\lambda,\ c = 4\lambda$
Put value of a, b and c in equaiotn 1
$18\lambda(x-2) + 17\lambda(y-1) + 4\lambda(z+1) = 0$
$18x + 17y + 4z = 49.$
Hence, vector equation of the plane is $\vec{r}.\ (18\hat{i} + 17\hat{j} + 4\hat{k}) = 49$.

Question: 22

The probabilities of two students A and B coming to the school in time are $\frac{3}{7}$ and $\frac{5}{7}$ respectively.

Assuming that the events. ‘A coming in time’ and ‘B coming in time’ is independent. Find the probability of only one of them coming to the school in time. Write at least one advantage of coming to school in time.

Answer:
Let probability that A comes to school on time = P (A)
Let probability that B comes to school on time = P (B)

$P(A) = \frac{3}{7}$

$P(B) = \frac{5}{7}$

OR

Find the vector equation of the plane through the points (2, 1,-1) and (-1, 3, 4) and perpendicular to the plane x -2y+ 4z = 10.

Answer:
Equation of plane passing through 2,1, 1 is
$a(x-2) + b(y-1) + c(z+1) = 0$ ……………equation (1)
This equation of plane pass through (-1,3,4).
Thus,
$a(-1-2) + b(3-1) + c(4+1) = 0$
$-3a+2b+5c = 0$………………………..equation (2)
Also, the above equation of plane is perpendicular to the plane x – 2y + 4 = 10
Thus,
A(1) +b9-2) + c(4) = 0
A – 2b + 4c = 0……………….equation (3)
Now, we have
$-3a + 2b + 5c = 0$
$a-2b + 4c = 0$
probability of only one of them coming to the school in time =

P(A) [1-P(B)] + P(B) [1-P(A)]

$$\frac{3}{7}.\frac{2}{7}+\frac{4}{7}.\frac{5}{7}$$

$$=\frac{6}{49}+\frac{20}{49}$$

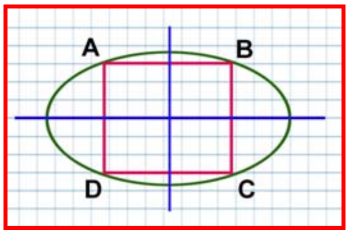
$$=\frac{26}{49}$$

Section C *(Question numbers 23 to 29 carry 6 marks each)*

Question: 23

Find the area of the greatest rectangle that can be inscribed in an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Answer:



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

The points are

A= (-a sinθ, b cosθ)

B = (a sinθ, b cosθ)

C = (-a sinθ, -b cosθ)

D = (-a sinθ, -b cosθ)

Thus,

AB = 2a sinθ

BC = 2b cosθ

Area of reactangle ABCD, S = AB X BC

S = (2ab sin2θ). (2bcosθ)

S = 2absin2θ

On differentiation

$$\frac{dS}{d\theta}=2ab\ (2\cos 2\theta)$$

$$\frac{dS}{d\theta}=4ab\cos 2\theta$$

$$\therefore \frac{dS}{d\theta}=0$$

4abcos2θ = 0

cos 2θ = 0

$$\theta=\frac{\pi}{4}$$

$$\frac{dS}{d\theta}=4ab\cos 2\theta$$

On further differentiation

$$\therefore \frac{d^2S}{d\theta^2}=8ab\sin 2\theta$$

$$\theta=\frac{\pi}{4}$$

$\therefore \frac{d^2S}{d\theta^2} = 8ab < 0$

$S = 2ab \sin 2\theta$

$S_{max} = 2ab\sin 2\left(\frac{\pi}{4}\right) = 2ab$

OR

Find the equations of tangents to the curve $3x^2 - y^2 = 8$, which pass through the Point $\left(\frac{4}{3}, 0\right)$.

Answer:

The equation of the curve is

$3x^2 - y^2 = 8$

On differentiation is

$6x - 2x.\ \frac{dy}{ds} = 0$

$\frac{dy}{ds} = \frac{3x}{y}$

Tangent at the point (x1,y1) on the curve is

$3x_1^{\ 2} - y_1^{\ 2} = 8$ ……………Equation(1)

Slope of the tangent = $\left.\frac{dy}{dx}\right|_{(x_1, y_1)} = \frac{3x_1}{y_1}$

Equation of the line passing through (x_1, y_1) with slope $\frac{3x_1}{y_1}$ is

$y - y_1 = \frac{3x_1}{y_1}(x - x_1)$

Point $\left(\frac{4}{3}, 0\right)$ lies on the above equation

$0 - y_1 = \frac{3x_1}{y_1}\left(\frac{4}{3}, x_1\right)$

$y_1^{\ 2} - 3x_1^{\ 2} + 4x_1 = 0$……………………….Equation(2)

Now, take equations 1 and 2

$3x_1^{\ 2} - y_1^{\ 2} = 8$

$y_1^{\ 2} = 3x_1^{\ 2} - 8$

$y_1^{\ 2} - 3x_1^{\ 2} + 4x_1 = 0$

$y_1^{\ 2} = 3x_1^{\ 2} - 4x_1$

Thus,

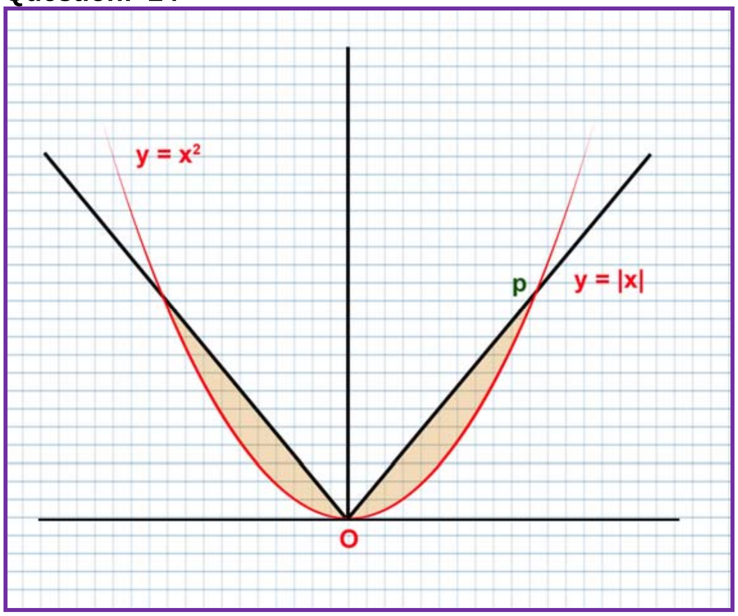
X1 = 2, y1 $\pm$ 2

Hence, equation of tangents are:

$y - 3x + 4 = 0$

$y + 3x - 4 = 0$

Question: 24



Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$ (**)

Question: 25

Find the particular solution of the differential equation $(\tan^{-1}y - x)\,dy = (1+y^2)dx$, given that when x = 0. y = 0. (**)

Question: 26

Find the equation of the plane passing through the line of intersection of $\vec{r}.(\hat{i}+3\hat{j})-6=0$ & $\vec{r}.(3\hat{i}-\hat{j}+4\hat{k})=0$ the planes whose perpendicular distance from origin is unity. (**)

Answer:

$(\tan^{-1}y-x)dy = (1+y^2)dx$

$$\frac{dx}{dy}=\frac{(\tan^{-1}y-x)}{(1+y^2)}$$

$$\frac{dx}{dy}=\frac{\tan^{-1}y}{1+y^2}-\frac{x}{1+y^2}$$

$$\frac{dx}{dy}+\frac{x}{1+y^2}=\frac{\tan^{-1}y}{1+y^2}$$

Integrating factor $= e^{\int\frac{dy}{1+y^2}} = e^{\tan^{-1}y}$

Now, multiply the above equation with integrating factor

$$e^{\tan^{-1}y}\frac{dx}{dy}+e^{\tan^{-1}y}.\frac{x}{1+y^2}=e^{\tan^{-1}y}.\frac{\tan^{-1}y}{1+y^2}$$

$$x.e^{\tan^{-1}y}=\int e^{\tan^{-1}y}.\frac{\tan^{-1}y}{1+y^2}.dy$$

Let $t = \tan^{-1}y$

$$dt = \frac{dy}{1+y^2}$$

$$x.e^{\tan^{-1}y} = \int e^t.tdt$$

Applying by parts

$$x.e^{\tan^{-1}y} = t(e^t) - \int e^t.\left\{\frac{d}{dt}(t)\right\}dt$$

$$x.e^{\tan^{-1}y} = t(e^t) - \int e^t.dt$$

$$x.e^{\tan^{-1}y} = te^t - e^t + C$$

$$x.e^{\tan^{-1}y} = \tan^{-1}y(e^{\tan^{-1}y}) - e^{\tan^{-1}y} + C$$

When x = 0, y = 0

$$0 = 0 = \tan^{-1}0(e^{\tan-10}) - e^{\tan-10} + C$$

0 = $-e^0$ +C

C = 1

Thus,

$$x.e^{\tan-1}y = \tan^{-1}y(e^{\tan-10}) + C$$

$$x = \tan^{-1}y - 1 + e^{-\tan^{-1}}y$$

Question: 27

In a hockey match, both teams A and B scored same number of goals up to the end of the game. So to decide the winner, the referee asked both the captains to throw a die alternately and decided that the team, whose captain gets a six first, will be declared the winner. If the captain of team A was asked to start, find their respective probabilities of winning the match and state whether the decision of the referee was fair or not.

Answer:

A starts the game

Probability of getting a six = $\frac{1}{6}$

Probability of not getting a six = $1-\frac{1}{6}=\frac{5}{6}$

Team A is declared, when captain of team A gets a six in first, third, fifth, seventh......;in these throws captain B will not get a six.

Team B is declared, when captain of team B gets a six in second, forth, sixth, eighth......;in these throws captain A will not get a six.

Probability that team A will be a winner = P(A)

Probability that team B will be a winner = P (B)

$$P(A) = \frac{1}{6} + \frac{5}{6}x\frac{5}{6}x\frac{1}{6} + \frac{5}{6}x\frac{5}{6}x\frac{5}{6}x\frac{5}{6}x\frac{1}{6} + ...\infty$$

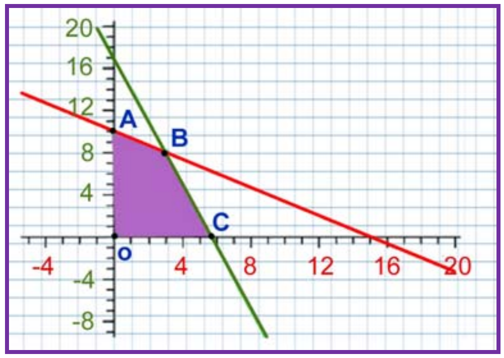
$$P(A) = \frac{1}{6}\left[1 + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^4 + ...\infty\right]$$

$$P(A) = \frac{1}{6}\left[\frac{1}{1-\left(\frac{5}{6}\right)^2}\right] = \frac{6}{11}$$

$$P(B) = \frac{5}{6}x\frac{1}{6} + \frac{5}{6}x\frac{5}{6}x\frac{5}{6}x\frac{1}{6}x\frac{5}{6}x\frac{5}{6}x\frac{5}{6}x\frac{1}{6}..........\infty$$

$$P(B) = \frac{5}{36}\left[\frac{1}{1-\left(\frac{5}{6}\right)^2}\right] = \frac{5}{11}$$

Question: 28



A manufacturer considers that men and women workers are equally efficient and so he pays them at the same rate. He has 30 and 17 units of workers (male and female) and capital respectively, which he uses to produce two types of goods A and B. To produce one unit of A, 2 workers and 3 units of capital are required while 3 workers and 1 unit of capital is required to produce one unit of B. If A & B are priced at Rs 100 and Rs 120 per unit respectively, how should he use his resources to maximise the total revenue? Form the above as an LPP and solve graphically. Do you agree with this view of the manufacturer that men and women workers are equally efficient and so should be paid at the same rate?

Answer:

Let the no. of units of

Let the no. of units of B

Total workers used in production of units of A & B 2x + 3y

Total capital used in production of units of A & B 3x + y

As per the question,

$2x + 3y \leq 30$

$3x + y \leq 17$

$Z = 100x + 120y$

$x \geq 0$

$y \geq 0$

Points	Z = 100x + 120y
A(0,10)	Z = 100(0) + 120(10) = 1200
B(3,8)	Z = 100(0) + 120(8) = 300 + 960
C $\left(\frac{17}{3}\right)$,0	Z = 100$\left(\frac{17}{3}\right)$+120(0) $\left(\frac{1700}{3}\right)$
O(0,0)	Z = 100(0) + 120(0) = 0

Question: 29

The management committee of a residential colony decided to award some of its members (say x) for honesty, some (say y) for helping others and some others (say z) for supervising the workers to keep the colony neat and clean. The sum of all the awardees is 12. Three times the sum of awardees for cooperation and supervision added to two times the number of awardees for honesty is 33. If the sum of the number of awardees for honesty and supervision is twice the number of awardees for helping others, using matrix method, find the number of awardees of each category. Apart from these values, namely, honesty, cooperation and supervision, suggest one more value which the management of the colony must include for awards.

Answer:
Let the awardees for honesty = x
Let the awardees for helping others = y
Let the awardees for keeping the colony neat & clean = z
As per the question,equations become as
X+y+z = 12
2x + 3y + 3z = 33
x-2y+z = 0

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 12 \\ 33 \\ 0 \end{bmatrix}$$

AX = B

$$A=\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 1 & -2 & 1 \end{bmatrix}$$

$$|A| = 1[3+6] – 1[2-3]+1[-4-3$$

$$|A| = 9+1-7$$

$$|A| = 3 \neq 0$$

$$\text{Adj}\,A=\begin{bmatrix} 9 & -3 & 0 \\ 1 & 0 & -1 \\ -7 & 3 & 1 \end{bmatrix}$$

$$A\text{-}1 = \frac{1}{3}\begin{bmatrix} 9 & -3 & 0 \\ 1 & 0 & -1 \\ -7 & 3 & 1 \end{bmatrix}$$

Now,
AX = B
$X = A^{-1}B$

$$X = \frac{1}{3}\begin{bmatrix} 9 & -3 & 0 \\ 1 & 0 & -1 \\ -7 & 3 & 1 \end{bmatrix}\begin{bmatrix} 12 \\ 33 \\ 0 \end{bmatrix}$$

$$X = \frac{1}{3}\begin{bmatrix} 108 & -99 & +0 \\ 12 & +0 & +0 \\ -84 & +99 & +0 \end{bmatrix}$$

$$X = \frac{1}{3}\begin{bmatrix} 9 \\ 12 \\ 15 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

X = 3, y = 4, z = 5
The awardees for honesty = 3
The awardees for helping others = 4
The awardees for keeping the colony neat and clean = 5

(**) Currently out of syllabus. Answer can be provided up on reques